Relativity, Mechanics and Quantum Theory

Problem Sheet 3

Problem 3.1

The affine group in D-dimensions is the group of rotations (by a D by D square matrix A) and translations by a D-dimensional vector b such that:

$$x \to x' = Ax + b$$

Find the composition law for two successive affine transformations and hence show that there is a one-to-one correspondence between this group and the group of $D+1$ by $D+1$ square matrices of the form:

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

(Note that these matrices are reducible but are only fully reducible if $b = 0$ in which case we have a representation of a compact rotational subgroup of the affine group rather than the full non-compact affine group.)

Problem 3.2

(a.) If $\Pi(g)$ is a finite-dimensional representation of a group G, show that the matrices $\Pi^*(g)$ also form a representation.

(b.) The representation $\Pi^*(g)$ may or may not be equivalent to $\Pi(g)$. If they are equivalent then there exists an intertwining map, T, such that:

$$\Pi^*(g) = T^{-1}\Pi(g)T$$

Show that, if $\Pi(g)$ is irreducible then $TT^* = \lambda\mathbb{I}$

(c.) If $\Pi(g)$ is unitary show that $TT^\dagger = \mu\mathbb{I}$. Show that T may be redefined so that $\mu = 1$ and that T is either symmetric or antisymmetric.

Problem 3.3

Let $\Pi_1 : G \to GL(V)$ and $\Pi_2 : G \to GL(W)$ be two irreducible representations of the group G on the finite-dimensional complex vector spaces V and$W$. Show that if we have two non-zero intertwining maps between Π_1 and Π_2 denoted T_1 and T_2 and taking $V \to W$ then $T_1 = \lambda T_2$ for some $\lambda \in \mathbb{C}$.