Relativity, Mechanics and Quantum Theory

Problem Sheet 2

Problem 2.1

The special unitary group of two-by-two square matrices is defined to be:

$$SU(2) \equiv \{M \in U(2) \,|\, \det(M) = 1\}$$

Where $U(2)$ indicates the unitary group of two-by-two square matrices.

(a.) Given that $SU(2)$ is three-dimensional, show that a general element of $SU(2)$ may be written in the form:

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

Where $\alpha, \beta \in \mathbb{C}$ and such that $|\alpha|^2 + |\beta|^2 = 1$.

(b.) Prove that $SU(2)$ is a group.

(c.) Find all elements contained in the centre of $SU(2)$. Show that it is isomorphic to the cyclic group $\mathbb{Z}_2$.

(d.) Show that the centre of $SU(2)$ is a normal subgroup of $SU(2)$: $Z(SU(2)) \triangleleft SU(2)$

(e.) Compute the conjugacy classes C_h of the elements

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$$

in $SU(2)$ (i.e. find the sets $C_h = \{ghg^{-1} \,|\, g \in SU(2)\}$). If one considers the elements of $SU(2)$ as points on a 3-sphere, S^3, what are the geometric shapes of these subsets in S^3?

Problem 2.2

If H is a normal subgroup of G show that the set of cosets $\frac{G}{H}$ is itself a group with the composition law:

$$(g_1 H) \cdot (g_2 H) = (g_1 \cdot g_2 H) \qquad \forall \quad g_1, g_2 \in G$$

This group is called the quotient group.

Problem 2.3

Let $\mathbb{Z}_p$ denote the cyclic group with p elements, i.e. $\mathbb{Z}_p = \frac{\mathbb{Z}}{p\mathbb{Z}}$. Show that there is no non-trivial homomorphism from $\mathbb{Z}_{p_1}$ to $\mathbb{Z}_{p_2}$ if p_1 and p_2 are different prime numbers. (Hint: One way to see this is to first show that $\mathbb{Z}_p$ has no non-trivial subgroups, using Lagranges theorem without proof. From there, argue that any non-trivial homomorphism between the two groups in question must be surjective. To proceed, you may use the fact that for any non-zero $n \in \mathbb{Z}_{p_1}$, we have $p_2 \cdot n \neq 0$, where $p_2 \cdot n$ means $n + \ldots + n$: adding up p_2 "copies of n.)

Problem 2.4

Consider $g_1, g_2 \in G$ where G is a group endowed with composition law $\circ$.

(a.) Give the definition of the conjugacy class C_g.

(b.) Show that two conjugacy classes C_{g_1} and C_{g_2} either coincide or are disjoint.

(c.) Show that if a conjugacy class has more than one element, then it cannot be a subgroup of G.