Relativity, Mechanics and Quantum Theory

Problem Sheet 1

Problem 1.1

(a.) D_n is the dihedral group the set of rotation symmetries of an n-polygon with undirected edges. Write down the multiplication table for D_3 defined on the elements {e,a,b} by $a^2 = b^3 = (ab)^2 = e$. Give a geometrical interpretation in terms of the transformations of an equilateral triangle for a and b.

(b.) Rewrite the group multiplication table of D_3 in terms of six disjoint cycles given by repeated action of the basis elements on the identity until they return to the identity, e.g. $e \to e$ under the action of e, $e \to a \to e$ under the action of a.

(c.) Label the vertices of the equilateral triangle by $(1, 2, 3)$. Denote the vertices of the triangle by $(1, 2, 3)$ and give permutations of $\{1, 2, 3\}$ for e, a and b which match the defining relations of D_3.

(d.) Rewrite each of the cycles of part (b.) in cyclic notation on the vertices $(1, 2, 3)$ to show this gives all the permutations of S_3.

Problem 1.2

Let $f : G \to G'$ be a homomorphism of groups. Show that $H \equiv \mathrm{Ker}(f) \equiv \{g \in G \,|\, f(g) = e'\}$ is a normal subgroup of G.
(Hint: Show that $gH = Hg$ for all $g \in G$. To do this show that both sides coincide with $f^{-1}(f(g))$, i.e. the set consisting of all those elements $\hat{g} \in G$ which satisfy $f(\hat{g}) = f(g)$.)

Problem 1.3

Consider the Klein four-group, V_4, (named after Felix Klein) consisting of the four elements $\{e, a, b, c\}$ and defined by the relations:

$$a^2 = b^2 = c^2 = e, \qquad ab = c, \qquad bc = a \qquad \text{and} \qquad ac = b$$

(a.) Show that V_4 is abelian.

(b.) Show that V_4 is isomorphic to the direct product of cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_2$. To do this choose a suitable basis of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and group composition rule and use it to show that the basis elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same relations as those of V_4.

(Definition: A group G is the direct product of subgroups A and B, written $G = A \times B$ if (i.) all the elements of A commute with those of B and (ii.) every element $g \in G$ can be written in a unique way as $g = ab$ where $a \in A$ and $b \in B$.)

Problem 1.4

Show that the alternating group A_n consisting of all the permutations $P \in S_n$ such that $\mathrm{Sign}(P) = 1$, is a normal subgroup of S_n. Use the group homomorphism $f : S_n \to \mathbb{Z}_2$. What is the order, $|A_n|$, of the alternating group?