conserved.

For example consider free circular motion (i.e. $V=0$):

$$\mathcal{L} = \frac{1}{2} m R^2 \dot{\theta}^2.$$

$\frac{\partial \mathcal{L}}{\partial \theta} = 0$, θ is ignorable hence $\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\dot{\theta}$ is conserved.

This is the conservation of angular momentum.

The quantity $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is called the generalised momentum and denoted p_i.

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Hamiltonians.

Hamiltonians also encode the dynamics of a physical system.

There is an invertible map from a Lagrangian to a Hamiltonian so no information is lost; the map is the Legendre transform.

$$H(q_i, p_i; t) = \sum_i \dot{q}_i p_i - \mathcal{L}.$$

Example.

Let $\mathcal{L} = \sum_i \frac{1}{2} m \dot{q}_i \dot{q}_i - V(q) \equiv T - V.$

Then $p^i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = m\dot{q}_i$

and
$$\begin{aligned} H &= \sum_i \dot{q}_i (m\dot{q}_i) - \sum_i \frac{1}{2} m \dot{q}_i^2 + V(q). \\ &= \frac{1}{2}\sum_i m\dot{q}_i^2 + V(q) \\ &(= T + V.) \\ &= \sum_i \frac{p_i^2}{2m} + V(q). \end{aligned}$$

N.B. The Hamiltonian is a function of q and p not q and $\dot{q}$.

The Hamiltonian is closely related to the total energy of the system.

While the dynamics of the Lagrangian were described in the n dimensional space of $q_i(t)$'s called configuration space. The dynamics of the Hamiltonian system are described on the $2n$-dimensional space of $(q_i(t), p_i(t))$ called phase space.

The trajectory/dynamics of an n-body system are given by a line in phase space.

<u>Hamilton's Equations.</u>

Now as $H \equiv H(q_i, p_i; t)$ then,

$$dH = \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t.$$

But as $H = \dot{q}_i p_i - \mathcal{L}$ we also have:

$$dH = \cancel{\delta \dot{q}_i p_i} + \dot{q}_i \delta p_i - \frac{\partial \mathcal{L}}{\partial q_i}\delta q_i - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial t}\delta t.$$

Comparing coefficients we have:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

The first two are commonly refered to as Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

These are $2n$ first order differential equations (cf. Lagrange's equations which are n 2nd order differential equations).

<u>Example.</u>

If $H = \frac{p^2}{2m} + V(q)$ then,

$$\dot{q} = \frac{\partial H}{\partial p} = p/m \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q} = -\partial V.$$

i.e. we find $p = \dot{q}m$ and N II : $F = -\partial V = \dot{p}$.

<u>Poisson Brackets.</u>

Observe that swapping $q_i \leftrightarrow p_i$ interchanges the pair of Hamilton's equations up to a sign. This kind of skew symmetry subtley indicates that Hamiltonian dynamical systems are related to symplectic manifolds $(Sp(2n))$. There is consequently a useful skew-symmetric structure on the phase space. It is called the Poisson bracket, it is given by:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

Where $f(q_i, p_i)$ and $g(q_i, p_i)$ are arbitrary functions on phase space.

The e.o.m. are then:

$$\{q_i, H\} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \{p_i, H\} = -\frac{\partial H}{\partial q_i}.$$

Furthermore note that:

$$\begin{aligned}\{f, H\} &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \\ &= \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) \\ &= \frac{\partial f}{\partial t}. \qquad \text{if } f(q_i, p_i)\end{aligned}$$

$$\therefore \dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}.$$

Example.

The Harmonic oscillator again. Recall that:

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

$$\therefore \quad H = \dot{q} p - \mathcal{L}$$

$$= p^2/m - p^2/2m + \frac{1}{2} k q^2$$

$$= p^2/2m + \frac{1}{2} k q^2.$$

And, $\quad \dot{q} = \frac{\partial H}{\partial p} = p/m$

while $\quad \dot{p} = -\frac{\partial H}{\partial q} = -kq.$

$$\therefore \quad \ddot{q} = \dot{p}/m$$

$$= -k/m\,(q)$$

$$\Rightarrow \quad q(t) = A\cos(\omega t) + B\sin(\omega t)$$

Where $\omega = \sqrt{k/m}$ is the frequency and A, B are real constants.

(Exactly as before.)

Noether's Theorem.

To every symmetry of the action there is an associated conserved quantity.

Let us denote the action $A_R[q]$ where,

$$A_R[q] \equiv \int_R dt\, \mathcal{L}(q, \dot{q}) \qquad R = [t_2, t_1]$$

There are two types of symmetry we would like to consider:

(i) Internal : $A_R[q'] = A_R[q]$

(ii) Space-time : $A_{R'}[q'] = A_R[q]$.

The internal symmetries.

Let $q_i \to q_i' = q_i + \varepsilon X_i(q)$ then,

$A_R[q_i + \varepsilon X_i(q)] = A_R[q_i]$ as $\delta A_R = 0$:

$$\delta A_R = A_R[q_i + \varepsilon X_i(q)] - A_R[q_i]$$
$$= A_R[q_i + \delta q_i] - A_R[q_i]$$
$$= \int_R dt \left(\mathcal{L}(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) - \mathcal{L}(q_i, \dot{q}_i) \right)$$

Now $\mathcal{L}(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) = \mathcal{L}(q_i, \dot{q}_i) + \delta q_i \frac{\partial \mathcal{L}}{\partial q_i} + \delta \dot{q}_i \frac{\delta \mathcal{L}}{\partial \dot{q}_i}$

$$\therefore \delta A_R = \int_R dt \left(\delta q_i \frac{\partial \mathcal{L}}{\partial q_i} + \delta \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$
$$= \int_R dt \left(\delta q_i \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right] \right) + \int_R \frac{d}{dt}\left[\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right] \cdot dt$$
$$= \int_R \frac{d}{dt}\left(\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) dt.$$

Since $q \to q'$ is a symmetry by definition then $\delta A_R = 0$

and so $$\frac{d}{dt}\left(\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = 0$$

$\therefore$ $X_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is a conserved quantity.

The external symmetries are characterised by an additional time translation:

$$q_i \to q_i' = q_i + \varepsilon X_i(q)$$

$$t \to t' = t + \varepsilon\, \xi(t)$$

Now for the action to be invariant under these transformations we have:

$$A_{R'}[q_i'] = A_R[q] \quad \text{with } R' \text{ is the image of } R$$

under the time-translation: $(\delta A_R[q] = 0)$.

$$\begin{aligned}
0 &= A_{R'}[q + \delta q] - A_R[q] \\
&= \left(A_{R'}[q + \delta q] - A_R[q + \delta q]\right) + \left(A_R[q + \delta q] - A_R[q]\right) \\
&= \int_{R'} dt'\, \mathcal{L}\left(q(t') + \delta q(t')\right) - A_R[q + \delta q] + \left(A_R[q + \delta q] - A_R[q]\right). \\
&= \int_R dt\, \mathcal{L}\left(q(t + \delta t) + \delta q(t + \delta t)\right)\left(1 + \varepsilon \frac{d\xi}{dt}\right) + \ldots
\end{aligned}$$

(as $dt' = dt + \varepsilon\, d\xi = dt + \varepsilon \frac{d\xi}{dt} \cdot dt = dt\left(1 + \varepsilon \frac{d\xi}{dt}\right)$).
and $\delta t \equiv \varepsilon\, \xi(t)$.

Note that $\mathcal{L}(q'(t + \delta t)) = \mathcal{L}(q'(t)) + \delta t \frac{\mathcal{L}}{dt} + O(\delta t^2)$.

Hence we have:

$$\begin{aligned}
0 &= \int_R dt \left[\varepsilon \xi \frac{\mathcal{L}}{dt} + \varepsilon \frac{d\xi}{dt} \mathcal{L}\right] + \int_R dt \left(\mathcal{L}(q') - \mathcal{L}(q)\right) + O(\delta t^2) \\
&= \int_R dt\, \frac{d}{dt}\left(\varepsilon \xi \mathcal{L}\right) + \frac{d}{dt}\left(x_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + O(\delta t^2) \\
&\quad + \int_R dt \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}_{0} \delta q_i.
\end{aligned}$$

Hence if additionally the action is invariant under time translation Q is a conserved charge where:

$$Q = \xi \mathcal{L} + x_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Examples.

1. Space translation is a symmetry $\Rightarrow$

$$q_i \to q_i + \varepsilon a_i$$

$$\therefore \quad Q = a_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv a_i p_i$$

The conserved charge is linear momentum.

2. Time translation is a symmetry of A $\Rightarrow$

$$t \to t + \varepsilon b \qquad b \text{ is a constant.}$$

and also $\quad q \to q(t - \varepsilon b) = q(t) - \varepsilon b \frac{dq}{dt} + O(\varepsilon^2)$.

$$\therefore \quad Q = b\mathcal{L} - b \frac{dq_i}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$= -b(\dot{q}_i p_i - \mathcal{L})$$

$$= -bE.$$

N.B. the energy function is the precursor to the Hamiltonian,

e.g. for $\quad \mathcal{L} = \frac{1}{2} m \dot{q}_i^2 - V(q)$

then $\quad \dot{q}p - \mathcal{L} = \dot{q}(m\dot{q}) - \frac{1}{2} m \dot{q}^2 + V(q)$

$$= \tfrac{1}{2} m \dot{q}^2 + V(q).$$

$$= E.$$

3. Let $\quad \mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$

let $\quad z = x + iy \quad \Rightarrow \quad \mathcal{L} = \frac{m}{2}(\dot{z}\dot{\bar{z}}) - \frac{k}{2}(z\bar{z})$

$$z \to z' = e^{i\omega} z = z + i\omega z + O(\omega^2) \quad \text{is a symmetry:}$$

$z\bar{z} \to z'\bar{z}' = e^{i\omega} z . e^{-i\omega} \bar{z} = z\bar{z}$ $\quad$ and similarly for $\dot{z}\dot{\bar{z}}$.

(ω is a constant).

$\left(\mathcal{L} = \frac{m}{2}\dot{z}\dot{\bar{z}} - \frac{k}{2}z\bar{z} \right).$

i.e. take $\delta z = i\omega z$ then
$\delta\bar{z} = -i\omega\bar{z}$

$$Q = iz\frac{\partial \mathcal{L}}{\partial \dot{z}} - i\bar{z}\frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}}$$

$$= iz\left(\frac{m}{2}\right)\dot{\bar{z}} - i\bar{z}\left(\frac{m}{2}\right)\dot{z}.$$

Check: $\frac{dQ}{dt} = \frac{m}{2}\left(i\cancel{\dot{z}\dot{\bar{z}}} + iz\ddot{\bar{z}} - i\cancel{\dot{\bar{z}}\dot{z}} - i\bar{z}\ddot{z}\right)$

$$= i\frac{m}{2}\left(z\ddot{\bar{z}} - \bar{z}\ddot{z}\right).$$

From the equations of motion we have:

z: $\frac{d}{dt}\left(\frac{m}{2}\dot{\bar{z}}\right) - \left(-\frac{k}{2}\bar{z}\right) = 0$

$$\Rightarrow \frac{m}{2}\ddot{\bar{z}} + \frac{k}{2}\bar{z} = 0.$$

$$\Rightarrow \ddot{\bar{z}} = -\frac{k}{m}\bar{z} \qquad \Rightarrow \ddot{z} = -\frac{k}{m}z.$$

$$\therefore \frac{dQ}{dt} = i\frac{m}{2}\left(-\frac{k}{m}z\bar{z} - \bar{z}\left(-\frac{k}{m}\right)z\right) = 0. //$$

Noether's theorem in the Hamiltonian formulation.

One may represent transformations on phase space using a generating function $f(q_i, p_i)$ by:

$$q_i \longrightarrow q_i' = q_i + \alpha\{q_i, f\}. \equiv q_i + \delta q_i.$$

and $$p_i \longrightarrow p_i' = p_i + \alpha\{p_i, f\}. \equiv p_i + \delta p_i.$$

Such a transformation where $\alpha << 1$ is called an infinitesimal canonical transformation: they preserve the form of Hamilton's equations in the new and old variables i.e.

$$\left.\begin{array}{l}\dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}\end{array}\right. \xrightarrow{\text{become}} \left.\begin{array}{l}\dot{q}' = \frac{\partial H'}{\partial p'}. \\ \dot{p}' = -\frac{\partial H'}{\partial q'}.\end{array}\right.$$

Such a transformation is a symmetry of the Hamiltonian if $\delta H = 0$ under the change of variables $q_i \to q_i'$, $p_i \to p_i'$.

Now,
$$\delta H = \sum_i \left(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right)$$
$$= \alpha \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$
$$= \alpha \{ H, f \}$$
$$= -\alpha \frac{\partial f}{\partial t}.$$

Hence if $\delta H = 0$ then f itself is conserved.

Quantum Mechanics.

Consider a classical system described by a Lagrangian:
$$\mathcal{L} = \sum_i \frac{m_i}{2} (\dot{q}_i)^2 - V(q_i) \qquad i = 1, \dots, n.$$

The equations of motion are:
$$m_i \ddot{q}_i + \frac{\partial}{\partial q_i}(V) = 0.$$
$$\Rightarrow \quad F = m_i \ddot{q}_i \qquad \text{(Newton's II law.)}$$

We also have the Hamiltonian:
$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$
$$= \frac{(p_i)^2}{2m} + V(q).$$

And the natural symplectic structure, the Poisson brackets:
$$\{ q_i, p_j \} = \delta_{ij}.$$

Quantisation is the promotion of the positions q_i and