Part II: Classical Mechanics and Quantum Mechanics.

2.1. Classical Mechanics.

Lagrangian Mechanics.

Newton's second law of motion states that for a body of mass m acted on by a force $\underline{F}$, its rate of change of linear momentum $\underline{p}$ is equal to the force $\underline{F}$:

$$\underline{F} = \frac{d}{dt}(\underline{p}) = \frac{d}{dt}(m\underline{\dot{x}}) = m\underline{\ddot{x}}$$

(For constant mass, m).

Hence if $\underline{F} = \underline{0}$ then, $\underline{\dot{p}} = 0$ and linear momentum is conserved.

$\underline{F}$ is called a conservative force if the energy $(E = \underset{\text{Kinetic energy.}}{T} + \underset{\text{Potential energy.}}{V})$ of the system is constant for motion under the force.

i.e. The work done moving on a path from $\underline{x}(t_1)$ to $\underline{x}(t_2)$ is:

$$\Delta W = \int_{t_1}^{t_2} dt \; \underline{F} \cdot \underline{\dot{x}} \qquad \left(\text{i.e. } \underline{F} \cdot \underline{\dot{x}}\, dt = \underline{F} \cdot d\underline{x}\right).$$

$$= \int_{t_1}^{t_2} dt \; m\underline{\ddot{x}} \cdot \underline{\dot{x}}$$

$$= \tfrac{1}{2} m \dot{x}^2(t_2) - \tfrac{1}{2} m \dot{x}^2(t_1).$$

$$= \Delta T \quad \text{the change in kinetic energy.}$$

In general the work done depends on the path taken:

Whenever the work done ΔW is path independent the force is called conservative. A conservative force can always be expressed as $\underline{F} = -\underline{\nabla} V$

Where V is a scalar function as in that case work done depends only on the values of V at the endpoints of the path:

$$\begin{aligned}
\Delta W &= \int_{t_1}^{t_2} \underline{F} \cdot d\underline{x} \\
&= \int_{t_1}^{t_2} -\underline{\nabla} V \cdot d\underline{x} \\
&= \int_{t_1}^{t_2} -\begin{pmatrix} \partial_x V \\ \partial_y V \\ \partial_z V \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\
&= -\int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) \\
&= -\int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial V}{\partial z} \frac{\partial z}{\partial t} \right) dt. \\
&= -\int_{t_1}^{t_2} \left(\frac{dV}{dt} \right) dt \\
&= -\left(V(t_2) - V(t_1) \right).
\end{aligned}$$

In terms of K.E. we also had $\Delta W = T(t_2) - T(t_1)$

$$\therefore \; T(t_2) - T(t_1) = V(t_1) - V(t_2)$$

$$\Rightarrow (T+V)(t_1) = (T+V)(t_2). //$$

Hence a conservative force conserves energy.

In terms of V, N II becomes:

$$-\partial_i V = m \ddot{x}_i$$

This law of motion may be derived from a variational principle on the functional:

$$A = \int_{t_1}^{t_2} dt\, \mathcal{L}.$$

called the action, where $\mathcal{L}$ is the Lagrangian.

$$\mathcal{L} = \mathcal{L}(\underline{x}, \dot{\underline{x}}; t). \equiv T - V$$

$$= \sum_{i=1}^{n} \tfrac{1}{2} m_i \dot{\underline{x}}_i^2 - \sum_{i=1}^{n} V_i .$$

For a system with n particles. (N.B. $V_i = V(\underline{x}_i)$.)

The equations of motion are found by extremizing A:

(For simplicity consider only a 1-body system: $n=1$)

$$\delta A = \int_{t_1}^{t_2} dt\, \delta \mathcal{L}$$

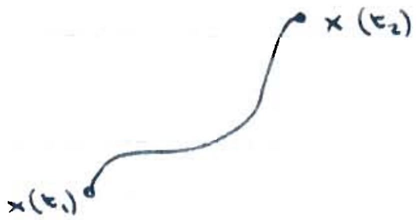
$$= \int_{t_1}^{t_2} dt\, \delta\left(\tfrac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x})\right)$$

$$= \int_{t_1}^{t_2} dt\, \left(m \dot{\underline{x}}\, \delta(\dot{\underline{x}}) - V(\delta \underline{x})\right)$$

$$= \int_{t_1}^{t_2} dt\, \left(m \dot{\underline{x}} \frac{d}{dt}(\delta \underline{x}) - \partial_i V(\delta x^i)\right)$$

$$= \int_{t_1}^{t_2} dt\, \left(- \frac{d}{dt}(m \dot{x}_i) - \partial_i V\right)(\delta x^i). + \left[\delta x^i m \dot{x}^i\right]_{t_1}^{t_2}.$$

To extremize we find the path $\underline{x}(t)$ such that $\delta A = 0$ for two fixed endpoints:



Since $\delta x(t_1) = \delta x(t_2) = 0$ we have the condition.

$$\delta A = 0 \implies -\frac{d}{dt}(m \dot{x}_i) - \partial_i V = 0.$$

Or: $-\partial_i V = m \ddot{x}_i$ for $\dot{m} = 0$.

Which is the Newtonian equation of motion.

More generally describe a generic system of particles by

n generalised coordinates : $q^1, \dots q^n$

n generalised velocities : $\dot{q}^1, \dots \dot{q}^n$.

The Lagrangian is then $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i; t)$.

The resulting equations of motion are called the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

Where for a single particle in $\mathbb{R}^3$ we have:

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V.$$

The generalised coordinates may be picked to be any n numbers which completely parameterise the resulting path of the particle.

E.g. for a free particle $V = 0$ (no force) and motion is in $\mathbb{R}^3$ so we need 3 parameters,

e.g. choose $q_1 = x$, $q_2 = y$, $q_3 = z$

then,

$$\mathcal{L} = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2).$$

And the Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

$$\Rightarrow \quad m\ddot{q}_i = 0.$$

Conservation of linear momentum.

Further examples.

1. Linear harmonic oscillator: $V(q) = \frac{1}{2}kq^2 \qquad k > 0.$

$$(\text{N.B.} \Rightarrow F = -kq)$$

$$\therefore \quad \mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2.$$

e.o.m. is $\quad \frac{d}{dt}\left(m\dot{q}\right) - kq = 0.$

$$\therefore \quad \ddot{q} = \frac{k}{m}q.$$

$$\Rightarrow \quad q = A\cos(\omega t) + B\sin(\omega t).$$

Where $\omega = \sqrt{\frac{k}{m}}$ is the frequency of oscillation and A and B are constants.

2. Circular motion:

Consider a bead of mass m constrained to move under gravity on a frictionless circular hoop of radius R that the hoop lies in a vertical plane.

The Lagrangian formulation offers a neat way to ignore forces of constraint via the choice of generalised coordinates.

If the hoop lies in the xz-plane centered at $z = R$ then:

$$x = R\cos\theta. \qquad \therefore \quad \dot{x} = -R\sin\theta.\dot{\theta}$$
$$y = 0. \qquad \dot{y} = 0.$$
$$z = R\sin\theta. + R. \qquad \dot{z} = R\cos\theta.\dot{\theta}.$$

Encodes the fact that the bead may only move on the hoop but without considering any of the forces acting to keep the bead on the hoop.

The generalised coordinates are simply $q = \theta$.

$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V$$

$V = mgz \implies -\partial_z V = F_G = -mg$ ✓

$$= \tfrac{1}{2}m(R^2\dot{\theta}^2) - mg(R\sin\theta + R).$$

$\therefore$ e.o.m. are: $\frac{d}{dt}(mR^2\dot{\theta}) - (mgR\cos\theta) = 0.$

$$\therefore \quad mR^2\ddot{\theta} = mgR\cos\theta.$$

$$\ddot{\theta} = \left(\frac{g}{R}\right)\cos\theta.$$

$$\therefore \quad \ddot{\theta} \doteq \left(\frac{g}{R}\right)\left(1 - \frac{\theta^2}{2} + O(\theta^4)\right)$$

For $\theta \ll 1$ then $\ddot{\theta} \simeq (g/R) \implies \theta \simeq \frac{1}{2}\left(\frac{g}{R}\right)t^2 + At + B.$

Conserved Quantities.

For every ignorable coordinate in a Lagrangian there is an associated conserved quantity.

E.g. if $\mathcal{L}(q_i, \dot{q}_i, t)$ satisfies $\frac{\partial \mathcal{L}}{\partial q_i} = 0$ (q_i is ignorable)

then $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$ by Lagrange's equation and $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is