Properties of the Lorentz Group.

- $\Lambda^T \cdot \eta \cdot \Lambda = \eta \Rightarrow \det(\Lambda) = \pm 1$.

and as $\eta_{00} = +1$ then $\Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 = +1$.

$$\Rightarrow \Lambda^0{}_0 \eta_{00} \Lambda^0{}_0 = +1.$$

$$\therefore (\Lambda^0{}_0)^2 = +1.$$

- The proper Lorentz group has $\det(\Lambda) = +1 \Rightarrow \Lambda \in SO(1,3)$.
- The orthochronous group has $(\Lambda^0{}_0) \geq 1$.

- $O(3)$ is a subgroup of $O(1,3)$ i.e. $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$ with $R \in O$

- In addition to ordinary rotations and reflections, we have "pure" Lorentz boosts: e.g.

$$\Lambda = \begin{pmatrix} \cosh(\xi) & -\sinh(\xi) & 0 & 0 \\ -\sinh(\xi) & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This indicates an x-direction boost, but equivalently there are boosts in the y and z directions.

(N.B. when a boost is combined with a rotation it is sometimes referred to as a boost too but the "pure" is dropped.)

- The proper orthochronous Lorentz group is generated completely by rotations and pure Lorentz boosts.

N.B. As there are 3 pure boosts and 3 independent rotations $SO(1,3)$ is generated by 6 continuous real parameters.

One can also see it has a basis of skew-symmetric matrices $\Rightarrow$ 6 parameters.

The proper Lorentz group and $SL(2,\mathbb{C})$.

Recall the Pauli matrices (and the identity):

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0. & 1. \\ 1. & 0. \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0. & -i. \\ i. & 0. \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1. & 0. \\ 0. & -1. \end{pmatrix}.$$

Now consider for each Lorentz vector x^μ the 2×2 matrix:

$$X \equiv x^\mu \sigma_\mu .$$

Now $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. and in fact $\sigma_i\sigma_j = \begin{cases} i\varepsilon_{ijk}\sigma_k. & i\neq j. \\ \delta_{ij}.\sigma_0. & i=j. \end{cases}$

And $X\sigma_\nu \equiv x^\mu\sigma_\mu\sigma_\nu = x^\mu i\varepsilon_{ijk}\sigma_k$.
or $x^\mu\delta_{\mu\nu}\sigma_0$.

So as $Tr(\sigma_1) = Tr(\sigma_2) = Tr(\sigma_3) = 0$ and $Tr(\sigma_0) = 2$.

$$Tr(X\sigma_\nu) = x_\nu Tr(\sigma_0) = 2x_\nu .$$

$$\therefore \quad x_\nu = \tfrac{1}{2} Tr(X\sigma_\nu).$$

Now $X \equiv x^\mu\sigma_\mu = \begin{pmatrix} x^0 - x^3. & -x^1 + ix^2. \\ -x^1 - ix^2. & x^0 + x^3. \end{pmatrix}$.

So that $\det(X) = (x^0)^2 - (x^3) - (x^1)^2 - (x^2)^2 = x^\mu x_\mu$.

$\therefore$ Transformations on X of the form:

$$A X A^\dagger = X'$$

Which preserve $\det(X) = X^2 = \det(X')$ represent a Lorentz transformation of x^μ. The condition on A is:

$$\det(A) = \det(A^\dagger) = 1.$$

Hence $A \in SL(2,\mathbb{C})$ encodes a proper Lorentz transformation on x^μ.

Generators of SO(3,1).

Using $\quad -iX = \left.\frac{d\Lambda}{d\theta}\right|_{\theta=0}.$

We have the rotation generators:

$$-iX_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0. \end{pmatrix}, \quad -iX_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad -iX_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

And the pure Lorentz boosts:

$$-iB_1 = \begin{pmatrix} 0. & -1. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}, \quad -iB_2 = \begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}, \quad -iB_3 = \begin{pmatrix} 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \end{pmatrix}.$$

Now $\quad [B_1, X_1] = 0.$

$\quad [B_1, X_2] = iB_3$

$\quad [B_1, B_2] = -iX_3. \qquad$ etc.

Giving: $\quad [X_i, X_j] = i\varepsilon_{ijk} X_k,$

$\quad [X_i, B_j] = i\varepsilon_{ijk} B_k,$

$\quad [B_i, B_j] = -i\varepsilon_{ijk} X_k.$

If one uses the combinations:

$$X_i^{\pm} \equiv \tfrac{1}{2}(X_i \pm iY_i) \qquad i = 1, 2, 3.$$

Then the commutation relations simplify:

$$[X_i^+, X_j^+] = i\varepsilon_{ijk} X_k^+ \qquad \text{su(2)}$$

$$[X_i^-, X_j^-] = i\varepsilon_{ijk} X_k^-. \qquad \text{su(2)}.$$

$$[X_i^+, X_j^-] = 0.$$

This gives two copies of the $SU(2)$ algebra commutators.
$$so(1,3) \otimes \mathbb{C} \cong su(2) \oplus su(2).$$
However locally $SO(1,3)$ is isomorphic to $SL(2,\mathbb{C})$

in fact the proper orthochronous Lorentz group is isomorphic

to $SL(2,\mathbb{C})/\mathbb{Z}_2$ (cf. $SO(3) \cong SU(2)/\mathbb{Z}_2$)

and $SL(2,\mathbb{C})$ is the complexification of $SU(2)$.

One can label the representations of $SO(1,3)$ by the

dimensions, or "spins", of the two $SU(2)$ irreps.

<u>The Poincaré Group.</u>

The Poincaré group consists of the Lorentz transformations

and additionally translations:

$$P \equiv \{ (\Lambda, a) \mid \Lambda \in \text{Lorentz group}, \ a \in \mathbb{R}^4 \}.$$

A general transformation of the Poincaré group takes the form:

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}.$$

Representations of the Lorentz Group and Lorentz Tensors.

- The trivial representation (objects which are invariant under the Lorentz transformations) such objects in the vector space of the representation are called Lorentz scalars.
- The standard representation on $\mathbb{R}^4$ (define it to be $\pi^{(1,0)}$) acts as: $x^{\mu} \xrightarrow{\Lambda} x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.

x^{ν} is a rank $(1,0)$ tensor component, also called contravariant.

- The contragredient representation acting on tensors with components x_μ:

$$x_\mu \to x'_\nu = \Lambda^\mu{}_\nu \, x_\mu .$$

x_μ are called co-vectors, or rank(0,1) tensors.

- In fact $\pi^{(0,1)}$ and $\pi^{(1,0)}$ are equivalent with the intertwiner map being η:

if $\pi^{(1,0)}(\theta) = \Lambda$ and $\pi^{(0,1)}(\theta) \equiv (\pi^{(1,0)}(\theta^{-1}))^T$

then as $\Lambda^{-1} = \eta \cdot \Lambda^T \cdot \eta$

$$\Rightarrow (\pi^{(0,1)}(\theta))^T = \eta \cdot (\pi^{(1,0)}(\theta))^T \cdot \eta^{-1}$$

$$\Rightarrow \pi^{(0,1)}\theta \equiv \eta \cdot \pi^{(1,0)}(\theta) \cdot \eta^{-1} .$$

- General tensor representations are built from products of the vector and co-vector representations and called (r,s)-tensors:

$$\underbrace{\pi^{(1,0)} \otimes \pi^{(1,0)} \otimes \ldots \otimes \pi^{(1,0)}}_{r.} \otimes \underbrace{\pi_{(0,1)} \otimes \ldots \otimes \pi_{(0,1)}}_{s.} .$$

(r,s)-tensors have components $T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}$.

And under a Lorentz transformation we have:

$$T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \xrightarrow{\Lambda} T'^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = \Lambda^{\mu_1}{}_{\kappa_1} \ldots \Lambda^{\mu_r}{}_{\kappa_r} \cdot \Lambda_{\nu_1}^{\,\,\,\,\rho_1} \ldots \Lambda_{\nu_s}^{\,\,\,\,\rho_s} \, T^{\kappa_1 \ldots \kappa_s}{}_{\rho_1 \ldots \rho_s} .$$

- $\eta_{\mu\nu}$ still acts to raise and lower indices:

$$\eta_{\mu_1 \rho} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = T_\rho{}^{\mu_2 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} .$$

- One can contract indices on an (r,s) tensor to obtain an $(r-1, s-1)$ tensor:

$$\eta_{\mu_r}{}^{\nu_s} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = T^{\mu_1 \dots \mu_{r-1}}{}_{\nu_1 \dots \nu_{s-1}}.$$

- One may be interested in special subsets of tensors:
  - Tensors with symmetric indices (e.g. consider an $(r,0)$ tensor):

$$T^{(\mu_1 \dots \mu_r)} \equiv \frac{1}{r!} \sum_{\pi \in S_r} T^{\mu_{\pi(1)} \mu_{\pi(2)} \dots \mu_{\pi(r)}}.$$

This is totally symmetric under the interchange of any two indices.

  - Anti-symmetric tensors:

$$T^{[\mu_1 \dots \mu_r]} \equiv \frac{1}{r!} \sum_{\pi \in S_r} \mathrm{Sign}(\pi)\, T^{\mu_{\pi(1)} \dots \mu_{\pi(r)}}.$$

This picks up a minus sign under the interchange of any two indices.

<u>Example $(r=2)$.</u>

$$T^{(\mu\nu)} \equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right) \qquad \therefore\ T^{(\mu\nu)} = T^{(\nu\mu)}.$$

$$T^{[\mu\nu]} \equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right) \qquad \therefore\ T^{[\mu\nu]} = -T^{[\nu\mu]}.$$

N.B. $\underbrace{T^{(\mu\nu)}}_{\text{Symmetric part.}} + \underbrace{T^{[\mu\nu]}}_{\text{Antisymmetric part.}} = T^{\mu\nu}.$

- Just as for $SO(3)$ we also have $\frac{1}{2}$ odd integer representations $\pi^{(1/2,0)}$ and $\pi^{(0,1/2)}$ these are the spinor representations of the Lorentz group.