1.4. Lorentz Tensors.

The Lorentz group, $O(1,3)$, is defined by:

$$O(1,3) \equiv \{ \Lambda, \text{ a } 4\times 4 \text{ matrix} \mid \Lambda^T \eta \Lambda = \eta \,;\, \eta = \mathrm{diag}(1,-1,-1,-1) \}.$$

The subgroup of proper Lorentz transformations has $\det = 1$ and is therefore $SO(1,3)$.

The matrix $\eta_{\mu\nu}$ defines the signature of an inner product.

For the orthonormal group one takes $\eta_{ij} = \delta_{ij}$.

The defining relation is an inner product on $\mathbb{R}^{p,q} \cong \mathbb{R}^{p+q}$. (Here $\mathbb{R}^{1,3}$).

Let v_μ be a four vector: $v = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$ and let $w = \begin{pmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{pmatrix}$

Then $\quad v^T \cdot \eta_{\mu\nu} \cdot w \equiv \langle v, w \rangle \quad$ defines an inner product.

$$\langle v, w \rangle = (v^0, v^1, v^2, v^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

$$= (v^0, v^1, v^2, v^3) \begin{pmatrix} w^0 \\ -w^1 \\ -w^2 \\ -w^3 \end{pmatrix}$$

$$= v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3$$

Since $\langle v, w \rangle$ is not positive definite this is not a scalar product. It is sometimes called a Minkowski product and $\mathbb{R}^4$ equipped with this inner product is frequently called Minkowski space. Notice that the length squared of any 4-vector on Minkowski space falls into one of three classes:

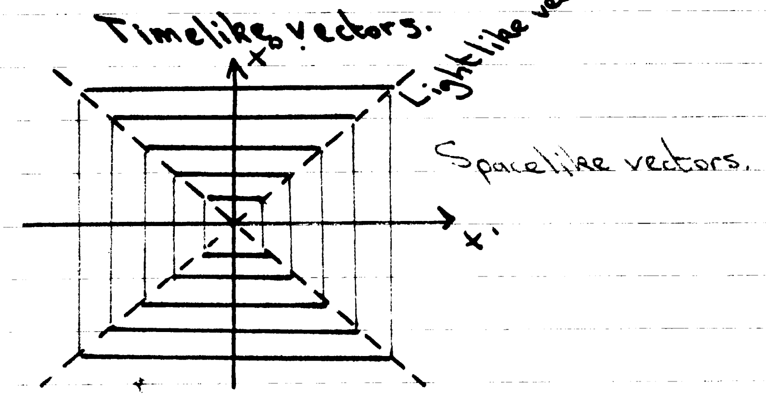
$\langle v, v \rangle > 0$ v is called timelike.

$\langle v, v \rangle = 0$ v is called lightlike.

$\langle v, v \rangle < 0$. v is called spacelike.

We will see the reason for this shortly.

Consider the subspace of $\mathbb{R}^4$ consisting of the x^0 and x^1 axes.



Let $v = \begin{pmatrix} v^0 \\ v^1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary vector in this space.

$$v^2 = \langle v, v \rangle = (v^0)^2 - (v^1)^2$$

Hence if

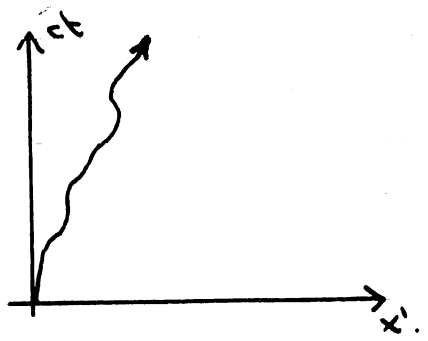
$v^0 > v^1$ v is timelike.
$v^0 = v^1$ v is lightlike.
$v^0 < v^1$ v is spacelike.

The main example of Minkowski space occurs in relativity where it is used to model spacetime with

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

Note the constant c is used for the speed of light so that the dimension of x^0 is metres and it is on the same footing as x^1, x^2, x^3.

If we plot the worldline for the one-dimensional motion of a particle, i.e. for a sequence of times t we measure x^1 and plot e.g.



What is the gradient of the worldline?

$$\text{Gradient} = \frac{\Delta(ct)}{\Delta(x^1)} = \frac{c\Delta t}{\Delta x^1} = c/\text{speed}.$$

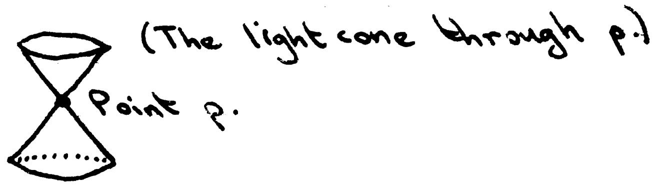
Hence if the particle moves at c then the gradient $= 1$.

In this case $x^2 = (x^0)^2 - (x^1)^2 = (ct)^2 - (x^1)^2 = 0$ (x is lightlike)

If gradient > 1 then speed $< c$ and x is timelike while if the gradient < 1 then speed $> c$ and x is spacelike.

Einstein's special theory of relativity postulated that speeds of physical objects did not cross the lightspeed barrier. Consequently any point in spacetime may only exchange information from its past and its future by a lightcone drawn from that point.

A light cone is the cone traced out by the directions of all possible lightlike vectors:



N.B. No spacelike seperated points can exchange a signal (since the message would have to travel at a speed exceeding that of light.)

The Lorentz transformations for a body moving at speed v with respect to the x-axis (but stationary with respect to y and z) are:

$$t' = \gamma\left(t - {xv}/{c^2}\right).$$

$$x' = \gamma(x - vt).$$

$$y' = y.$$

$$z' = z.$$

Where $\gamma = \left(1 - {v^2}/{c^2}\right)^{-1/2}$.

N.B. As $v \to 0$ $t' \to t$, $x' \to x$ but as $v \to c$ $t' \to \infty$, $x' \to \infty$.

Notice that under a Lorentz transformation $\langle \underline{x}, \underline{x} \rangle$ is invariant:

$$\begin{aligned}
\langle \underline{x}', \underline{x}' \rangle &= (ct')^2 - (x')^2 - (y')^2 - (z')^2 \\
&= \left(c^2\gamma^2\left(t^2 + {x^2v^2}/{c^4} - 2t\cancel{{xv}/{c^2}}\right)\right) - \gamma^2(x^2 + v^2t^2 - 2\cancel{x}vt) - y^2 - z^2. \\
&= t^2(c^2\gamma^2 - v^2\gamma^2) - x^2\left(\gamma^2 - {c^2\gamma^2v^2}/{c^4}\right) - y^2 - z^2. \\
&= c^2t^2(c^2 - v^2)(c^2 - v^2)^{-1} - x^2\left(\frac{c^2 - v^2}{c^2}\right)^{-1}\left(1 - {v^2}/{c^2}\right) - y^2 - z^2. \\
&= (ct)^2 - x^2 - y^2 - z^2 \\
&= \langle \underline{x}, \underline{x} \rangle
\end{aligned}$$

The invariance of the inner product is our first hint that Lorentz transformations are related to $O(1,3)$.

We can rewrite the Lorentz boost (i.e. we did not consider a rotation) so that it looks a little like a rotation.

Let $\gamma \equiv \cosh(\xi)$

Now $\cosh^2(\xi) - \sinh^2(\xi) = 1 = \gamma^2 - \sinh^2(\xi)$

i.e. $\sinh^2(\xi) = \gamma^2 - 1$.

$\therefore \quad \tanh(\xi) = \frac{1}{\gamma}(\gamma^2 - 1)^{1/2} = (1 - 1/\gamma^2)^{1/2} = (1 - (1 - v^2/c^2))^{1/2} = v/c.$

Hence,

$$t' = \cosh(\xi)\left(t - \frac{x}{c}\tanh(\xi)\right) = \cosh(\xi)\, t - \frac{x}{c}\sinh(\xi)$$

$$x' = \cosh(\xi)\left(x - ct.\tanh(\xi)\right) = \cosh(\xi)\, x - ct\sinh(\xi).$$

Which one may think of as a hyperbolic rotation of x and ct.

We have been discussing the position 4-vector $x_\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ and we now introduce the index notation that hitherto we have made implicit use of. We denote the 4-vector by x_μ with components: $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$.

We write its length squared by using the Einstein summation convention (every repeated index is summed over: $x^\mu x_\mu \equiv \sum_\mu x^\mu x_\mu$.)

$$x^2 = \langle \underline{x}, \underline{x} \rangle \equiv (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2$$
$$= x_\nu x_\mu \eta^{\mu\nu} = x^\nu x^\mu \eta_{\mu\nu} = x^\nu x_\mu \eta^\mu{}_\nu \quad \ldots$$

Let us think slowly about indices. Let $A \in M(4,\mathbb{R})$ and denote by x^μ a column 4-vector and by $A^\mu{}_\nu$ the component in the μ-th column and ν-th row.

Now since the transpose of a column vector is a row vector we denote $(x^\mu)^T = x_\mu$ * (using a "downstairs" index to denote a row entry).

Thus $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ and $x_\mu = (x^0 .\ x^1 .\ x^2 .\ x^3)$

Hence in this notation

$$\Big(\sum_\mu x_\mu x^\mu \equiv \Big) x_\mu x^\mu = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3$$

Einstein Summation Convention.

$$= (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 .$$

We have re-discovered the scalar product!

How do we write matrix multiplication?

$\underline{\underline{A}} \cdot \underline{x}$ sums the product of the rows of $\underline{\underline{A}}$ with the columns of $\underline{x}$

i.e. $$A^\mu{}_\nu x^\nu = A^\mu{}_0 x^0 + A^\mu{}_1 x^1 + A^\mu{}_2 x^2 + A^\mu{}_3 x^3 \equiv x'^\mu$$

Where x'^μ is the transformed vector under $\underline{\underline{A}}$.

Note the matching of free indices on both sides of the equation and also how the index positions are matched due to our conventions.

We may use the Minkowski metric $\eta_{\mu\nu}$ to lower indices

* We use the convention $(A^\mu{}_\nu)^T \equiv A^\nu{}_\mu = A_\mu{}^\nu$ - we simply swap up and down.

rather than the Euclidean transpose we used earlier.

We write $\eta_{\mu\nu}$ as a matrix (even though it defies our conventions):

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Then we define its matrix multiplication on x^{μ} as the lowering of its index:

$$\eta_{\mu\nu} x^{\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}^{T} \equiv x_{\mu}.$$

(N.B. $(x_{\mu})^{T} = (\eta_{\mu\nu} x^{\nu})^{T} = x_{\nu} \eta^{\nu\mu} = x^{\mu}$.)

Now we have $x^{\mu} x^{\nu} \eta_{\mu\nu} = x^{\mu} x_{\mu} = x^0(x^0) + x^1(-x^1) + x^2(-x^2) + x^3(-x^3)$

$$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$$= \langle \underline{x}, \underline{x} \rangle.$$

For $\Lambda \in O(1,3)$ we have:

$$\Lambda^{T} \cdot \eta \cdot \Lambda = \eta$$

(N.B. we would have usually: $(\Lambda^{T})^{i}{}_{j}\, \eta^{j}{}_{k}\, \Lambda^{k}{}_{l} = \Lambda^{j}{}_{i}\, \eta^{j}{}_{k}\, \Lambda^{k}{}_{l}$

But for η we write both indices down: $\Lambda^{j}{}_{i}\, \eta_{jk}\, \Lambda^{k}{}_{l}$.

$\equiv \Lambda^{T} \cdot \eta \cdot \Lambda.$)

Or,

$$(\Lambda^{T})^{\rho}{}_{\mu} \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = \Lambda^{\mu}{}_{\rho}\, \eta_{\mu\nu}\, \Lambda^{\nu}{}_{\sigma} \quad (= \Lambda^{\mu}{}_{\rho}\, \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma}) \quad = \eta_{\rho\sigma}.$$

Notice that $\eta^{-1} = \eta$ but we write the inverse matrix with upstairs indices so that $\eta^{-1} \cdot \eta = \eta^{\mu\nu} \cdot \eta_{\nu\rho}$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = \delta^{\mu}{}_{\rho}.$$

We can raise indices with $\eta^{\mu\nu}$: $\eta^{\mu\nu} x_{\nu} = x^{\mu}$.

Note also that since the components are numbers the component equations: $\eta_{\mu\nu} x^\nu = x^\nu \eta_{\mu\nu}$ the order is irrelevant.

Hence $x_\mu \cdot A^\mu{}_\nu = x^T \cdot A = A^\mu{}_\nu x_\mu$.

We are also free to raise and lower simultaneously pairs of indices which are to be summed over.

As,

$$x_\mu x^\mu = x^\nu \eta_{\mu\nu} x^\mu = x^\nu x_\nu = x^\mu x_\mu .$$

Hence,

$$x^T \cdot A = x_\mu \cdot A^\mu{}_\nu = x^\mu A_{\mu\nu} = A_{\mu\nu} x^\mu$$
$$= A_{\mu\nu} \eta^{\mu\rho} \eta_{\sigma\mu} x^\mu \delta^\sigma{}_\rho$$
$$= A^\rho{}_\nu x_\sigma \delta^\sigma{}_\rho$$
$$= A^\rho{}_\nu x_\rho .$$

Let us consider two events, one occurring at 4-vector $y^\mu \equiv \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and the second at $x^\mu \equiv \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

In Newtonian physics the difference in the time $(t_2 - t_1)$ between the two events and the distance between them $|(x^i - y^i)|$ $(i = 1, 2, 3;\ i \neq 0)$ would be invariant. As we have seen, under the Lorentz transformation, a new invariant emerges: $|x^\mu - y^\mu|^2 \equiv c^2 \tau_{xy}^2$ which is called the proper time τ.

i.e. $c^2 \tau_{xy}^2 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$.

One can also check that the Minkowski product $\langle x^\mu, y^\mu \rangle$

i.e. $\langle x^\mu, y^\mu \rangle = c^2 t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$

is also left unchanged by the Lorentz transformations.

These are called Lorentz-invariant quantities.

4-vectors all transform in the same way as x^μ under a Lorentz transformation (just as 3D vectors all transform in the same way under SO(3) rotations. We can find other physically relevant four-vectors by combining x^μ with Lorentz invariant quantities,

e.g. $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \cdot \frac{dt}{d\tau} = 1/\left(\frac{d\tau}{dt}\right) \cdot \begin{pmatrix} c \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is 4-velocity

Where $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is the usual spatial velocity vector and

$$\tau = \frac{1}{c}\left(c^2 t^2 - x^2 - y^2 - z^2\right)^{1/2}.$$

i.e. $\frac{d\tau}{dt} = \frac{1}{c}\,\frac{1}{2}\left(1/c\tau\right)\left(2c^2 t - 2x u_1 - 2y u_2 - 2z u_3\right)$

$$= \left(t - \tfrac{x}{c^2}u_1 - \tfrac{y}{c^2}u_2 - \tfrac{z}{c^2}u_3\right)/\tau.$$
$$= t\left(1 - u^2/c^2\right)/\left(t^2 - x^2/c^2 - y^2/c^2 - z^2/c^2\right)^{1/2} = (1/\gamma^2)\,\gamma = 1/\gamma.$$

We can check that u^2 is invariant:

$$u^\mu = \gamma\begin{pmatrix} c \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow u_\mu u^\mu = \gamma^2\left(c^2 - u_1^2 - u_2^2 - u_3^2\right)$$

$$= \gamma^2 c^2\left(1 - u^2/c^2\right)$$

$$= c^2. //$$

The 4-momentum $p^\mu = m u^\mu$

$$= m\gamma\left(c, u_1, u_2, u_3\right).$$

The spatial part is proportional to the Newtonian momentum p_N

$$p_N = m\underline{u}$$

And

$$(p^1, p^2, p^3) = \gamma(p_N^1, p_N^2, p_N^3).$$

While the zeroth component is identified with energy, E, via:

$$p^0 = E/_c = \gamma mc.$$

The invariant quantity associated to p^μ is:

$$p^\mu p_\mu = (E/_c)^2 - \gamma^2 p_N^2. = m^2c^2.$$

Let $\hat{p} = \gamma p_N$ and then,

$$(E/_c)^2 - \hat{p}^2 = m^2c^2.$$

$$\Rightarrow \quad (E/_c) = (m^2c^2 + \hat{p}^2)^{1/2}$$

Which is the Einsteinian version of $E = \frac{1}{2}m\underline{v}^2 = p^2/_{2m}$.

For a particle at rest $\hat{p} = \gamma(0)\, p_N = 1.0 = 0.$

$$\therefore \quad E/_c = mc \quad \Rightarrow \quad E = mc^2. //$$

N.B. we can also rewrite Lorentz invariant quantities using the index notation and see quickly that they are indeed invariants:

$$\text{e.g. } \eta_{\mu\nu} x'^\mu y'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma y^\sigma$$

$$= \Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma x^\rho y^\sigma.$$

$$= \eta_{\rho\sigma} x^\rho y^\sigma. //$$