1.3. Lie Groups.

Many of the groups we have met have been parameterised by discrete variables e.g. $\{e, g, g^2\}$ for $\mathbb{Z}_3$ but frequently a number have been described by continuous parameters such as $SO(n)$, $SU(n)$, $U(n)$, $Sp(n)$. For example $SO(2)$ describing rotations of $\mathbb{R}^2$ is parameterised by θ which takes values in the continuous set $[0, 2\pi)$ (for example) for each value of θ we find an element of $SO(2)$:

$$R(\theta) = \begin{pmatrix} \cos\theta. & -\sin\theta. \\ \sin\theta. & \cos\theta. \end{pmatrix}$$

(We may check that $R(\theta) R^T(\theta) = \mathbb{1}$).
and $\det(R(\theta)) = 1$.

In the language of representation theory $R(\theta)$ is a two-dimensional representation of the abstract group $SO(2)$. We may however check that it is a faithful representation of $SO(2)$, $R(0) = \mathbb{1}$, and the Kernel is trivial for $\theta \in [0, 2\pi)$.

The two-dimensional representation is irreducible over $\mathbb{R}$ but it is reducible over $\mathbb{C}$.

Over $\mathbb{C}$ we take as column vector $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} x + iy \\ x - iy \end{pmatrix} = \begin{pmatrix} re^{i\theta} \\ re^{-i\theta} \end{pmatrix}$.

and an $SO(2)$ rotation takes $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} \to \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} r e^{i(\theta+\phi)} \\ r e^{-i(\theta+\phi)} \end{pmatrix}$.

$$\Rightarrow \quad R(\phi, \mathbb{C}) = \begin{pmatrix} e^{i\phi}. & 0. \\ 0. & e^{-i\phi} \end{pmatrix}.$$

There is a qualitative difference when we move from $\mathbb{R}$ to $\mathbb{C}$ as this matrix is block diagonal and hence evidently reducible, into two 1D complex representations each of $U(1)$:

i.e. $\pi_m(\theta) = e^{im\theta}$ $\quad m \in \mathbb{Z}$ so that $\pi_m(\theta+2\pi) = \pi_m(\theta)$.

One can also check the decomposable nature of $SO(2,\mathbb{C})$ using the characters. Now since the group is compact $\frac{1}{|G|}\sum_g \to \int_0^{2\pi} \frac{d\theta}{2\pi}$

so that, $\langle \chi_m, \chi_n \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{im\theta} e^{-in\theta} = \delta_{mn}$.

Let $\pi_m \otimes \pi_n = \sum_\mu a_\mu \pi_\mu$.

Then $a_\nu = \langle \chi_{\pi_\nu}, \chi_{\pi_m \otimes \pi_n} \rangle$

For a 1D representation $\chi_{\pi_m} = e^{im\theta}$ the group element.

And $\chi_{\pi_m \otimes \pi_n} = \chi_{\pi_m} \cdot \chi_{\pi_n} = e^{i(m+n)\theta}$.

$$\begin{aligned}
\therefore \quad a_\nu &= \langle \chi_{\pi_\nu}, \chi_{\pi_m \otimes \pi_n} \rangle \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\nu\theta} \cdot e^{-i(m+n)\theta} \\
&= \delta_{\nu,(m+n)}
\end{aligned}$$

$\therefore \pi_m \otimes \pi_n = \pi_{m+n}$. //

We can also decompose the 2D representation $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Let $R(\theta) = \sum_\mu b_\mu \pi_\mu$

$$\begin{aligned}
\text{then} \quad b_\mu &= \langle \chi_{\pi_\mu}, \mathrm{Tr}(R(\theta)) \rangle \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi} \cdot e^{i\mu\theta} \cdot 2\cos\theta.
\end{aligned}$$

$$b_n = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{in\theta}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$= \delta_{n,1} + \delta_{n,-1}$$

$$\therefore\ R(\theta) = \pi_1(\theta) + \pi_{-1}(\theta). \quad //$$

As we used earlier.

We may identify $SO(2)$ with $U(1)$ and more geometrically the circle, S^1.

For other continuous groups we may also make an identification with a geometry:

- for $\{\mathbb{R}\setminus\{0\}, \cdot\}$ $\quad \cdots$ ———∘——— $\cdots$ two open half-lines.
- $SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\} = S^3$, as a set.

The generalised notion for a geometric setting is the manifold, each of the above are manifolds. Any geometric space one can imagine can be embedded in some Euclidean $\mathbb{R}^n$ as a surface of some dimension $\leq n$. For example the circle $S^1 \subset \mathbb{R}^2$, or more generally $S^{n-1} \subset \mathbb{R}^n$. No matter how extraordinary the curvature of the surface (so long as it remains well-defined) a manifold will have the appearance of being a Euclidean space at a sufficiently local scale. Consider $S^1 \subset \mathbb{R}^2$ sufficiently close to a point in S^1 the segment of S^1 appears identical to $\mathbb{R}^1$.

The geometry of a manifold is found by piecing together these open and locally-Euclidean sets. Each open neighbourhood is called a chart and is equipped with a map ϕ that converts points $p \in \mathcal{M}$ the manifold to local Euclidean coordinates. Using these local coordinates one can carry out all the usual mathematics in $\mathbb{R}^n$. The global structure of a manifold is defined by how these open sets are glued together. Since a manifold is a very well defined structure these transition functions are defined to be smooth. The study of manifolds, is the beginning of learning about differential geometry and is the subject of another course on the MSc so we will not dwell on the matter here except to make two points.

- As a point $p \in \mathcal{M}$, a manifold, varies it moves continuously so manifolds seem a good setting for the parameters labelling continuous groups.
- We have already observed that some of the compact matrix groups have geometrical interpretations. We may well wonder if this is always true for any continuous group.

~~Before giving the definitive response to these ponderings let~~

~~Since $A \in O(n)$ satisfy $\det A = \pm 1$ no such matrices exist forming a continuous path from A to B. A similar argument can be made for $GL(n, \mathbb{R})$ splitting it into components with $\det > 0$ and $\det < 0$.~~

Infinitesimal Generators.

The generators of a Lie group are introduced by considering elements infinitesimally close to the identity.

For $SO(2)$ with $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ we can expand:

$$R(\theta) = \mathbb{1} - i\theta X + O(\theta^2).$$ (Physics conventions!)

Or equivalently:

$$-iX = \left.\frac{dR(\theta)}{d\theta}\right|_{\theta=0}.$$

(i.e. the first term in the Taylor expansion of $R(\theta)$ about $R(0)$).

Specifically,

$$-iX = \left.\begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix X is called the infinitesimal generator of $SO(2)$.

N.B. X is skew-symmetric:

$$\mathbb{1} = R(\theta)R^T(\theta) = (\mathbb{1} - i\theta X + O(\theta^2))(\mathbb{1} - i\theta X^T + O(\theta^2))$$

$$= 1 - i\theta(X + X^T) + O(\theta^2)$$

Hence $X = -X^T$.

Also $\quad 1 = \det(R(\theta)) = \det(\mathbb{1} - i\theta X + O(\theta^2))$

$$\simeq 1 + i\theta \, \mathrm{Tr}(X) + O(\theta^2).$$

Hence $\mathrm{Tr}(x) = 0$.

One can quickly learn properties of the generators by imposing the definitions of the group.

Now the subgroup G_0 of all the group elements continuously connected to $\mathbb{1}$ can be constructed by exponentiation of X

i.e. $\quad \exp(-i\theta X) \in G_0$

One can check for $SO(2)$:

$$\exp(-i\theta X) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\theta)^n X^n$$

Now $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n = \Bigg\{ \underbrace{\overset{\mathbb{1}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}_{n=0}, \underbrace{\overset{X}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}_{n=1}, \underbrace{\overset{-\mathbb{1}}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}}_{n=2}, \underbrace{\overset{-X}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}_{n=3}, \underbrace{\overset{\mathbb{1}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}_{n=4}, \overset{\dots}{\dots} \Bigg\}.$

$$\therefore \quad \exp(-i\theta X) = \mathbb{1} - i\theta X + \tfrac{1}{2}\theta^2 \mathbb{1} - \tfrac{1}{3!}\theta^3 X + \tfrac{1}{4!}\theta^4 \mathbb{1} + \dots$$

$$= \mathbb{1}\left(1 + \tfrac{1}{2}\theta^2 + \tfrac{1}{4!}\theta^4 + \dots\right) - iX\left(\theta + \tfrac{1}{3!}\theta^3 + \dots\right)$$

$$= \mathbb{1}\cos\theta - iX\sin\theta.$$

$$= R(\theta). \;//$$

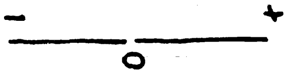
<u>Def</u>

__Def$^{\underline{n}}$__: For each Lie group G $\exists$ a vector space called the <u>Lie algebra</u> of G, denoted $\mathfrak{g}$ or $\mathrm{Lie}(G)$, such that for $\forall\ g \in G_0$ there is $X \in \mathrm{Lie}(G)$ with $e^{itX} = g$. $\ \forall\ t \in \mathbb{R}$.

Examples.

- $z \in U(1)$ then $\exists\ x \in \mathbb{R}$ s.t. $z = e^{ix}$.

- $\mathbb{R} \setminus \{0\}$ consists of two disconnected components:



The +ve half-line contains the multiplicative identity: 1; this is G_0.

It can be written as $\exp(x)$ where $x \in \mathbb{R}$.

- $SU(2)$ elements may be written as:

$$M = \exp(i\,\underline{\alpha}\cdot\underline{\sigma}) \quad \text{where} \quad \underline{\alpha} \in \mathbb{R}^3 \quad \text{and} \quad \underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

is a vector with matrix entries where:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are called the Pauli matrices.

Let us check this assertion. Let $\underline{\alpha} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then,

$$\exp(i\underline{\alpha}\cdot\underline{\sigma}) = \exp\begin{pmatrix} ic & ia+b \\ ia-b & -ic \end{pmatrix} \equiv \exp(iY).$$

$$Y^0 = \mathbb{1}, \quad Y = \begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}, \quad Y^2 = \begin{pmatrix} c^2+a^2+b^2 & 0 \\ 0 & a^2+b^2+c^2 \end{pmatrix} = (a^2+b^2+c^2)\mathbb{1}.$$

$$Y^3 = (a^2+b^2+c^2)Y, \ \ldots$$

$$\therefore \exp(iY) = \mathbb{1} + iY - \frac{1}{2}Y^2 - \frac{i}{3!}Y^3 + \frac{1}{4!}Y^4 + \ldots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ic & ia+b \\ ia-b & -ic \end{pmatrix} + \frac{1}{2}(a^2+b^2+c^2)\mathbb{1} + \frac{1}{3!}(-i)(a^2+b^2+c^2)Y + \ldots$$

Let $\alpha^2 \equiv a^2+b^2+c^2$

$$\therefore \exp(iY) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + iY \quad \frac{1}{2}\alpha^2(\mathbb{1}) - i\frac{1}{3!}\alpha^2 Y + \frac{1}{4!}\alpha^4 + \ldots$$

$$\exp(iY) = \mathbb{1}\left(1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{4!}\alpha^4 + \dots\right) + i\frac{Y}{\alpha}\left(\alpha - \tfrac{1}{3!}\alpha^3 + \dots\right)$$

$$= \mathbb{1}\left(\cos(\alpha)\right) + \frac{i}{\alpha}Y\left(\sin(\alpha)\right).$$

N.B. $\det Y = -c^2 - a^2 - b^2 = -\alpha^2 \Rightarrow \det(Y/\alpha) = -1.$

$\therefore +iY/\alpha$ is a matrix preserving area and we may write:

$$+iY/\alpha = \begin{pmatrix} in_3 & in_1 + n_2 \\ in_1 - n_2 & -in_3 \end{pmatrix} \quad \text{s.t. } \det\left(\frac{iY}{\alpha}\right) = n_3^2 + n_2^2 + n_1^2 = 1.$$

Giving:

$$\exp(iY) = \begin{pmatrix} \cos\alpha + in_3\sin\alpha & in_1\sin\alpha + n_2\sin\alpha \\ in_1\sin\alpha - n_2\sin\alpha & \cos\alpha - in_3\sin\alpha \end{pmatrix} = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}$$

$\in SU(2).$

- $SO(3)$ can be constructed from the rotations around the x, y, z axes namely:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence the infinitesimal generators are:

$$-iX \equiv \left.\frac{dR_x(\theta)}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

$$-iY \equiv \left.\frac{dR_y(\theta)}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow Y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$-iZ \equiv \left.\frac{dR_z(\theta)}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem. Let G be a matrix Lie group and let X be an element of $Lie(G)$ (or $\mathfrak{g}$) and let $A \in G$ then $AXA^{-1} \in Lie(G)$.

Proof. $$e^{it(AXA^{-1})} = \mathbb{1} + (it)(AXA^{-1}) + \frac{1}{2!}(it)^2(AXA^{-1})^2 + \ldots$$

But $(AXA^{-1})^n = AX^nA^{-1}$

Hence, $$e^{it(AXA^{-1})} = A\left(\mathbb{1} + itX + \frac{1}{2!}(it)^2X^2 + \ldots\right)A^{-1}$$
$$= Ae^{itX}A^{-1} \in G$$

$$\Rightarrow AXA^{-1} \in Lie(G).$$

N.B. $\frac{d}{dt}\left(e^{it(AXA^{-1})}\right)\Big|_{t=0} = iAXA^{-1}$. Hence to find $Lie(G)$ we use $-i\frac{d}{dt}\cdot\big|_{t=0}$.

Note that this defines a map on the Lie algebra acting as $G \times Lie(\mathfrak{g}) \to Lie(\mathfrak{g})$. The group elements act by conjugation on the Lie algebra to give an action moving through the algebra. Since $A \in G$ we may write $A = e^{itY}$ for some element $Y \in Lie(G)$ and find this action of the group on the algebra directly in the algebra itself.

The action moves from $X \to e^{itY}Xe^{-itY} \in Lie(G)$

Which is a path $y(t) \equiv e^{itY}Xe^{-itY}$ through the algebra parameterised by t. Let us look at the infinitesimal action:

$$-i\frac{d}{dt}(y(t))\Big|_{t=0} = -i\left((iY)e^{itY}Xe^{-itY} + e^{itY}X(-iY)e^{-itY}\right)\Big|_{t=0}.$$
$$= YX - XY.$$
$$\equiv [Y, X]. \quad \in Lie(G).$$

N.B. $e^{itX} \in G \Rightarrow X = -i\frac{d}{dt}(e^{itX})\big|_{t=0} \in Lie(G)$

Hence, in terms of matrix Lie groups, we have discovered the Lie bracket:

<u>Def$^{\underline{n}}$</u> Given any two $n \times n$ matrices A and B the <u>Lie bracket</u> (or commutator) $[A,B]$ is defined to be:

$$[A,B] = AB - BA.$$

This motivates a second definition of a Lie algebra:

<u>Def$^{\underline{n}}$.</u> A <u>Lie algebra</u> $(V, [\,,\,])$ is a vector space V together with a bilinear map $[\,,\,]: V \times V \to V$ called the Lie bracket such that $\forall\ u, v, w \in V$:

(ANTISYMMETRY) (i.) $[u, v] = -[v, u]$

(JACOBI IDENTITY) (ii.) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$

By linearity if the Lie brackets are known on the basis elements of $\mathrm{Lie}(G)$ then they may be found for the full $\mathrm{Lie}(G)$.

If $\{X_a\} \in \mathrm{Lie}(G)$ are a basis then,

$$[X_a, X_b] = \sum_c f_{ab}^{\ \ c} X_c$$

Where $f_{ab}^{\ \ c} = -f_{ba}^{\ \ c}$ are called the structure constants.

<u>Examples.</u>

- The Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

form a basis of $\mathrm{Lie}(SU(2))$, we find the commutators are:

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= 2i\,\sigma_3 .$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= -2i\,\sigma_2 .$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$= 2i\sigma_1 .$$

$$\therefore \quad [\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

N.B. one can choose a different basis to make the structure constants real e.g. $\left[\left(\frac{-i\sigma_i}{2}\right), \left(\frac{-i\sigma_j}{2}\right)\right] = \varepsilon_{ijk}\left(\frac{-i\sigma_k}{2}\right)$.

- Let us recall the Lie algebra basis for SO(3):

$$-iX_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad -iX_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad -iX_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Gives the commutation relations:

$$-[X_1, X_2] = -iX_3, \quad -[X_1, X_3] = iX_2, \quad -[X_2, X_3] = -iX_1$$

i.e. $[X_i, X_j] = i\,\varepsilon_{ijk} X_k$.

Hence we see an algebra isomorphism: $X_i \leftrightarrow \frac{\sigma_i}{2}$ between Lie(SU(2)) and Lie(SO(3