Proof:

Part (i.) That $T = 0$ or is an isomorphism.

Ker (T) and Im(T) are invariant subspaces, as

If $v \in \text{Ker}(T)$ then $T.v = \{\underline{0}\}$.

$$\therefore T.\pi_1(g).v = \pi_2(g)T(v) = \{\underline{0}\}.$$

$$\Rightarrow \pi_1(g)\, v \subset \text{Ker}(T) \quad \forall\, v \in \text{Ker}(T).$$

Hence Ker(T) is an invariant subspace of $\pi_1(g)$. //

As $\pi_1(g)$ is an irreducible representation of G Ker$(T) = \{\underline{0}\}$ or V.

If Ker$(T) = V$ then T is a map sending all $v \in V$ to $\underline{0} \in W$.

If Ker$(T) = \{\underline{0}\}$ the map is more interesting: immediately we can say that T is injective. The question remains whether $T(V)$ is a subspace of W or if T is surjective. Next observe that as T is injective $\exists\, v$ s.t. $T(v) = w \in W$

$$\therefore \quad T(\pi_1(g)(v)) = \pi_2(g)T(v) = \pi_2(g).w \in W. \quad \forall\, g \in G.$$

$\therefore T(V)$ is an invariant subspace of $\pi_2(g)$ in W.

As $\pi_2(g)$ is an irreducible representation it has no invariant subspaces apart from $\{\underline{0}\}$ and W itself. Hence Im(T) is either $\{\underline{0}\}$ or W. As we may presume V to be nontrivial and T is injective then Im$(T) = W$ and T is surjective. Hence T is an isomorphism. //

Part (ii.) If T is an isomorphism such that $T: V \to V$ and V is finite-dimensional over $\mathbb{C}$. Then:

$$T\pi(g) = \pi(g)T.$$

As the vector space is over the complex field $\mathbb{C}$, an eigenvalue always exists: $\quad T v = \lambda v \quad \Rightarrow \det(T - \lambda \mathbb{1}) = 0$.

This gives a polynomial in λ which always has a solution over $\mathbb{C}$.

Now if v is an eigenvector of T we have:

$$T . \pi(g)(v) = \pi(g) T(v) = \lambda \pi(g)(v).$$

Hence $\pi(g)(v)$ is another eigenvector with eigenvalue λ.

Hence the "λ-eigenspace" is an invariant subspace of π.

As π is an irreducible representation the λ-eigenspace is either $\{\underline{0}\}$ or V and since at least one eigenvalue exists the λ-eigenspace is not zero but equals V.

$$\therefore \; T v = \lambda v \qquad \forall \; v \in V.$$

$\Rightarrow \quad T \equiv \lambda \mathbb{1}$. As required.

A corollary of Schur's lemma is that if we have two intertwining maps T_1 and T_2 both acting to take $V \to W$ which are both non-zero then $T_1 = \lambda T_2$ for some $\lambda \in \mathbb{C}$.

E.g. If $T_2 \neq 0$ then it is an isomorphism, with the inverse map T_2^{-1} also an intertwiner. Now $T_1 \cdot T_2^{-1} \pi = \pi T_1 \cdot T_2^{-1}$ so $T_1 \cdot T_2^{-1}$ is an intertwiner taking $W \to W$.

$\therefore$ By Schur's lemma: $\quad T_1 \cdot T_2^{-1} = \lambda \mathbb{1} \Rightarrow T_1 = \lambda T_2$.

Def^n. Let V be a vector space endowed with the scalar product $\langle\, ,\, \rangle$. A representation $\pi : G \to GL(V)$ is called <u>unitary</u> if $\pi(g)$ are unitary operators i.e. if

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \qquad \forall g \in G,\ v, w \in V.$$

Def^n. If $\pi : G \to GL(V)$ is a representation on a finite dimensional vector space V the <u>character</u> of π is the function $\chi_\pi : G \to \mathbb{C}$ by

$$\chi_\pi(g) = Tr(\pi(g)).$$

Properties of the character:

- $\chi_\pi(e) = Tr(\pi(e)) = Tr(\mathbb{1}) = \dim(V).$
- $$\chi_\pi(ghg^{-1}) = Tr(\pi(g \cdot h \cdot g^{-1})) = Tr(\pi(g) \cdot \pi(h) \cdot \pi(g^{-1}))$$
$$= Tr(\pi(h))$$
$$= \chi_\pi(h).$$

Hence χ_π is constant on the conjugacy classes of G.

Such a function is called a class function.

- If π is unitary then

$$\chi_\pi(g^{-1}) = Tr(\pi(g^{-1}))$$
$$= Tr((\pi(g))^{-1})$$
$$= Tr(\pi^*(g))$$
$$= \chi_{\pi^*}(g) = \overline{\chi_\pi(g)}.$$

* If π_1 and π_2 are equivalent then $\chi_{\pi_1}(g) = \chi_{\pi_2}(g)$.

i.e.
$$\begin{aligned} \chi_{\pi_1}(g) &= Tr(\pi_1(g)) \\ &= Tr(T^{-1}\pi_2(g)T) \\ &= Tr(\pi_2(g)) \\ &= \chi_{\pi_2}(g). \end{aligned}$$

N.B. $T\pi_1 = \pi_2 T$
$\therefore \pi_1 = T^{-1}\pi_2 T$.

* If two representations of G have the same character they are equivalent.

<u>The direct sum and tensor product.</u>

Given two representation $\pi_{1/2}: G \to GL(V)$ we can form two important representations:

The direct sum: $\pi_1 \oplus \pi_2 : G \to GL(V_1 \oplus V_2)$

such that $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g)$.

N.B. $V_1 \oplus V_2$ has basis $\{\underline{e}_1, \underline{e}_2, \dots \underline{e}_n, \underline{f}_1, \underline{f}_2, \dots \underline{f}_m\}$.

if V_1 has basis $\{\underline{e}_1, \dots, \underline{e}_n\}$ and

V_2 has basis $\{\underline{f}_1, \dots, \underline{f}_m\}$.

and hence has dimension $\dim(V_1) + \dim(V_2) = n+m$.

The tensor product: $\pi_1 \otimes \pi_2 : G \to GL(V_1 \otimes V_2)$

such that $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$.

N.B. $V_1 \otimes V_2$ has basis $\{ e_i \otimes f_j \mid i = 1 \dots n, j = 1 \dots m\}$.

hence has dimension $\dim(V_1) \times \dim(V_2) = n.m$.

One can introduce scalar products and character functions on the direct and tensor product spaces as:

$$\langle v \oplus w, v' \oplus w' \rangle_{V \oplus W} \equiv \langle v, v' \rangle_V + \langle w, w' \rangle_W$$

and $$\langle v \otimes w, v' \otimes w' \rangle_{V \otimes W} \equiv \langle v, v' \rangle_V \cdot \langle w, w' \rangle_W .$$

$$\chi_{\pi_1 \oplus \pi_2}(g) = Tr(\pi_1(g)) + Tr(\pi_2(g))$$

$$\chi_{\pi_1 \otimes \pi_2}(g) = Tr_V(\pi_1(g)) \cdot Tr_W(\pi_2(g)).$$

You might think that all the information about these product representations is contained already in V and W. However consider the endomorphisms (the homomorphisms from a vector space to itself - if invertible then these are called the automorphisms):

Any $A \in End(V \oplus W)$ may be written:

$$A = \begin{pmatrix} A_{VV}. & A_{VW}. \\ A_{WV}. & A_{WW}. \end{pmatrix} \quad \text{where } A_{VV}: V \to V, \; A_{VW}: W \to V \text{ etc.}$$

i.e. $A_{VV} \in End(V)$ and $A_{WW} \in End(W)$ do not generate all the endomorphisms of $V \oplus W$ $\big(\dim(End(V \oplus W)) = (n+m)^2$
$\geq \dim(End(V)) + \dim(End(W))$
$= n^2 + m^2.$ //

On the other hand $A_{VV} \otimes A_{WW} \in End(V \otimes W)$ and the endomorphisms of V and W do generate all the endomorphisms of $V \otimes W$ $\big(\dim(End(V \otimes W)) = n^2 m^2 = (\dim(End(V))) \cdot (\dim(End(W)))$.

One can check that $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ are homomorphisms:

$$(\pi_1 \oplus \pi_2)(g_1 \cdot g_2) = \begin{pmatrix} \pi_1(g_1 g_2). & 0. \\ 0. & \pi_2(g_1 g_2). \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1(g_1)\cdot \pi_1(g_2). & 0. \\ 0. & \pi_2(g_1)\cdot \pi_2(g_2) \end{pmatrix}.$$

$$= \begin{pmatrix} \pi_1(g_1). & 0. \\ 0. & \pi_2(g_1). \end{pmatrix} \cdot \begin{pmatrix} \pi_1(g_2). & 0. \\ 0. & \pi_2(g_2) \end{pmatrix}.$$

$$= (\pi_1 \oplus \pi_2)(g_1) \cdot (\pi_1 \oplus \pi_2)(g_2).$$

$$(\pi_1 \otimes \pi_2)(g_1 \cdot g_2) = \pi_1(g_1 \cdot g_2) \otimes \pi_2(g_1 \cdot g_2)$$

$$= \pi_1(g_1) \cdot \pi_1(g_2) \otimes \pi_2(g_1) \cdot \pi_2(g_2).$$

$$= (\pi_1 \otimes \pi_2)(g_1) \cdot (\pi_1 \otimes \pi_2)(g_2).$$

The direct sum never gives an irreducible representation, having two non-trivial subspaces: $V \oplus 0 \cong V$ and $0 \oplus W \cong W$.

It is less straightforward with the tensor product to discover whether or not the representation is irreducible.

Frequently one can be interested in decomposing the tensor product into direct sums of irreducible sub representations:

$$V \otimes W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$$

To do this one must find an endomorphism of $V \otimes W$ (or change of basis) such that:

$$T(\pi_1 \otimes \pi_2(g))T^{-1} = \hat{\pi}_1(g) \oplus \ldots \oplus \hat{\pi}_n(g). \qquad T \in \mathrm{End}(V \otimes W).$$

The decomposition $\pi(G) \otimes \pi(G) = \sum a_i \pi_i(G)$ is called a Clebsch-Gordan decomposition.

This is not always possible. One can achieve this decomposition for one example crucial to quantum mechanics: $G = SU(2)$.

It is a fact that $SU(2)$ has only one unitary irreducible representation for each $\dim(V)$ vector space (up to equivalence).

More precisely there is an $(n+1)$-dimensional representation which may be better known due to $SU(2)/\mathbb{Z}_2 \cong SO(3)$ as the irreducible representations of angular momentum in quantum mechanics which are labelled by the spin j.

i.e. $SU(2)$ irreps labelled by $\dim(V) = n+1$

$SO(3)$ irreps labelled by spin j.

And in fact $j = n/2$, hence as $n \in \mathbb{Z}^+$ $j \in \mathbb{Z}^+/2$ and may be $1/2$-integer. N.B. Bosons: $j \in \mathbb{Z}^+$
Fermions: j takes fractional values $(\frac{1}{2}, \frac{3}{2}, \ldots)$.

$j = 0 \Rightarrow n = 0 \Rightarrow \dim(V) = 1$ is the "trivial" representation of $SU(2)$.
$j = \frac{1}{2} \Rightarrow n = 1 \Rightarrow \dim(V) = 2$ is the "standard" or "fundamental" representation of $SU(2)$.
$j = 1 \Rightarrow n = 2 \Rightarrow \dim(V) = 3$ is the "adjoint" representation of $SU(2)$.

The fusion rules of $SU(2)$ list the equivalence classes into which the tensor product of two $SU(2)$ irreps $[j_1]$ and $[j_2]$ decompose:

$$[j_1] \otimes [j_2] = [j_1 + j_2] \oplus [j_1 + j_2 - 1] \oplus \ldots \oplus [|j_1 - j_2|].$$

Some examples: $[0] \otimes [j] = [j]$
Dim: $1 \times (2j+1) = 2j+1$.

$$[\tfrac{1}{2}] \otimes [j] = [\tfrac{1}{2} + j] \oplus [-\tfrac{1}{2} + j].$$

Dim: $(2(\tfrac{1}{2})+1) \cdot (2j+1) \qquad (1+2j)+1 \; + \; (-1+2j)+1$

$= 4j+2. \qquad\qquad = 2+2j+2j$

$= 4j+2.$

N.B. $[\tfrac{1}{2}] \otimes [\tfrac{1}{2}] = [1] \oplus [0].$

$2 \cdot 2 \qquad 3 + 1.$

$[1] \otimes [\tfrac{1}{2}] = [\tfrac{3}{2}] \oplus [\tfrac{1}{2}].$

$3 \cdot 2. \qquad 4 + 2.$ etc.

So for $SU(2)$ one can generate all the irreps from tensor products of $[\tfrac{1}{2}]$ with itself. Hence $[\tfrac{1}{2}]$ is the fundamental representation.

We will return to $SU(2)$ when we discuss quantum mechanics.

For other groups the decomposition theory is more involved: to work out the Clebsch-Gordan coefficients one must know the inequivalent irreducible representations of G, its conjugacy classes and its character table (as well as a number of useful identities on characters, principally the orthogonality relation.).

If a representation of a group itself may be rewritten as a sum of representations it is by definition not irreducible, or reducible.

<u>Def$^{\underline{n}}$</u> A representation of dimension $n+m$, $\pi(g)$, $g \in G$ is reducible if $\pi(g)$ has the form:

$$\pi(g) = \begin{pmatrix} A(g). & C(g). \\ 0. & B(g). \end{pmatrix} \qquad \forall \; g \in G$$

A is $n \times n$, B is $m \times m$, C is $n \times m$ and 0 is the empty $m \times n$ matrix.

If we multiply two such matrices together we have:

$$\pi(g_1)\,\pi(g_2) = \begin{pmatrix} A(g_1). & C(g_1). \\ 0. & B(g_1). \end{pmatrix} \cdot \begin{pmatrix} A(g_2). & C(g_2). \\ 0. & B(g_2). \end{pmatrix}$$

$$= \begin{pmatrix} A(g_1)A(g_2). & A(g_1)C(g_2) + C(g_1)B(g_2). \\ 0. & B(g_1)B(g_2). \end{pmatrix}$$

$$= \pi(g_1 \cdot g_2)$$

$$\equiv \begin{pmatrix} A(g_1 g_2). & C(g_1 g_2). \\ 0. & B(g_1 g_2). \end{pmatrix}$$

Hence $A(g_1)A(g_2) = A(g_1 g_2)$

and $B(g_1)B(g_2) = B(g_1 g_2)$

So A and B are both representations of G but with dimensions n and m respectively.

For finite groups C can be shown to be equivalent to the null matrix (see Maschke's theorem). In this case the representation is said to be completely reducible (or decomposable) and:

"All reducible representations of a finite group are completely reducible."

$$\pi(g) = A(g) + B(g).$$

It does not follow that $A(g)$ and $B(g)$ themselves are reducible, but they may be and if so the process may be repeated until $\pi(g)$ is expressed as a direct sum of irreducible representations.