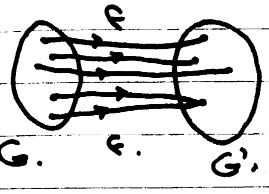
In fact if $f: G \to G'$ is surjective:

e.g.



then $G/_{Ker(f)} \cong G'$.

N.B. This is known as the isomorphism theorem and can be stated as $Im(f) \cong G/_{Ker(f)}$. where $f: G \to G'$ is a homomorphism.

We associate the $Im(f) \equiv f(g)$ to the coset $g\,Ker(f)$.

i.e. $f(g) \leftrightarrow g\,Ker(f)$.

Let us check this is well-defined. We would be concerned that two or more elements of G are mapped to one element of G'.

Suppose $f(g_1) = f(g_2)$ their images appear to be associated to two different cosets $g_1 Ker(f)$ and $g_2 Ker(f)$. However

since $f(g_1) = f(g_2)$ then
$$f(g_1) \circ' f(g_2^{-1}) = f(g_1 \circ g_2^{-1}) = f(g_2 \circ g_2^{-1}) = e'.$$

$\therefore (g_1 \circ g_2^{-1}) \in Ker(f)$ and so:

$$g_2 Ker(f) = Ker(f)\, g_2 = Ker(f) \circ (g_1 \circ g_2^{-1}) \circ g_2$$
(as $Ker(f) \triangleleft G$)
$$= Ker(f) \circ g_1$$
$$= g_1 Ker(f) \;//$$

The reverse association is also well-defined $g\,Ker(f) \mapsto f(g)$.

Again if g_1 and g_2 are two representatives of $g\,Ker(f)$

then $(g_1^{-1} \circ g_2) \in Ker(f)$ and so:

$$g_1 Ker(f) \mapsto f(g_1) \qquad g_2 Ker(f) \mapsto f(g_2)$$

But

$$\begin{array}{ccccc} g_1 Ker(f) & = & g_1 \circ (g_1^{-1} \circ g_2) Ker(f) & = & g_2 Ker(f) \\ \downarrow & & & & \downarrow \\ f(g_1) & & & & f(g_2). \end{array}$$

Hence all is well: $f(g_1) = f(g_2)$.

We have convinced ourselves that the mapping and its inverse is injective (1-to-1),

but is it a group homomorphism?

$$\begin{array}{ccc} f(g_1) \circ' f(g_2), & \longleftrightarrow & (g_1 Ker(f)) \circ (g_2 Ker(f)) \\ \shortparallel & & \shortparallel \\ f(g_1 \circ g_2) & \longleftrightarrow & g_1 g_2 Ker(f). \end{array}$$

By the composition law for coset groups.

(We could be more explicit and define the map $\phi : f(g) \to g\,Ker(f)$

then above we have shown that ϕ, ϕ^{-1} are surjective and injective

(hence ϕ is bijective) and since ϕ and ϕ^{-1} are group

homomorphisms ϕ is an isomorphism.)

ϕ is a homomorphism:

$$\begin{aligned} \phi(f(g_1) \circ f(g_2)) &= \phi(f(g_1 \circ g_2)) \\ &= g_1 g_2 Ker(f) \\ &= g_1 Ker(f) \circ' g_2 Ker(f) \\ &= \phi(f(g_1)) \circ' \phi(f(g_2)). \end{aligned}$$

ϕ^{-1} is a homomorphism:

$$\begin{aligned} \phi^{-1}(g_1 Ker(f) \circ' g_2 Ker(f)) &= \phi^{-1}(g_1 g_2 Ker(f)) \\ &= f(g_1 \circ g_2) \\ &= f(g_1) \circ' f(g_2) = \phi^{-1}(g_1 Ker(f)) \circ \phi^{-1}(g_2 Ker(f)). \end{aligned}$$

N.B. the method for finding new groups is essentially the same as looking for normal subgroups but one has the benefit of knowing that $Ker(f)$ is a normal subgroup. It is interesting when f is surjective only onto an unknown subgroup of G'.

3. One can form the direct product of groups to create more complicated groups. The direct product of G and H is denoted $G \times H$ with composition law:

$$(g_1, h_1) \circ' (g_2, h_2) \equiv (g_1 \circ g_2, h_1 \circ h_2).$$

Where $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

N.B. the direct product of $G \times G$ has a natural subgroup $\Delta(G)$ called the diagonal $\Delta(G) \equiv \{(g,g) \in G \times G \mid g \in G\}$.

4. If X is a set and G a group such that $\exists\ f : X \to G$ then the functions f with the composition rule

$$f_1 \circ' f_2 \ (x) \equiv f_1(x) \circ f_2(x)$$

G composition law.

E.g. if $X = S^1$ the set of maps of X into G form the "loop group" of G.

N.B. the identity map, maps $f(x) \to e \in G$.

There are only a finite number of finite simple groups.

The quest to identify them all is universally accepted as having been completed in the 1980's. In addition to groups such as the cyclic groups $\mathbb{Z}_n$, the symmetric group S_n, the dihedral group D_n and the alternating group A_n there are 14 other infinite series and 26 other sporadic groups. These include:

The Mathieu groups $(|M_{24}| = 2^{10}\cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244{,}823{,}040)$

The Janko groups $(|J_4| \simeq 8.67\times 10^{19})$.

The Conway groups $(|Co_1| \simeq 4.16\times 10^{18})$.

The Fischer groups $(|Fi_{24}| \simeq 1.26\times 10^{24})$.

The Monster group $(|M| \simeq 8.08\times 10^{53})$.

Def$^{\underline{n}}$. Let G be a group and X a set. An (left) action of G on X (or operation) is a map $G\times X \to X$ denoted

$$(g,x) \mapsto g\cdot x \equiv T_g(x)$$

Satisfying:

(i.) $(g_1 \cdot g_2)\cdot x = g_1 \cdot (g_2 \cdot x)$ $\quad g_1, g_2 \in G, x \in X$

(ii.) $e \cdot x = x \quad \forall\, x \in X$.

The set X is called a (left) G-set.

Def$^{\underline{n}}$. The orbit of $x\in X$ under the G-action is

$$G\cdot x \equiv \{ x' \in X \mid x' = g\cdot x \;\forall\, g\in G\}.$$

Def$^{\underline{n}}$. The __stabiliser subgroup__ of $x \in X$ is the group of all $g \in G$ such that $g \cdot x = x$ i.e.

$$G_x \equiv \{ g \in G \mid g \cdot x = x \}.$$

Def$^{\underline{n}}$. The __fundamental domain__ is the subset $X_F \subset X$ such that

$$\text{(i.)} \quad x \in X_F \implies g \cdot x \notin X_F \qquad \forall\, g \in G \setminus \{e\}.$$

$$\text{(ii.)} \quad X = \bigcup_{g \in G} g \cdot X_F.$$

__Examples.__

1. S_n acts on the set $\{1, 2, \ldots, n\}$.

2. G can act on itself in three canonical ways:

   (i.) Left translation: $T_{g_1}(g_2) = g_1 \cdot g_2$.

   (ii.) Right translation: $T_{g_1}(g_2) = g_2 \cdot g_1^{-1}$

   (iii.) The conjugate action : $T_{g_1}(g_2) = g_1 \cdot g_2 \cdot g_1^{-1}$
   or conjugation $\equiv Ad_{g_1}(g_2)$.
   or the Adjoint action

N.B. if L_g indicates left translation and R_g right translation

then: $Ad_{g_1}(g_2) = R_{g_1}^{-1} \cdot L_{g_1} \cdot g_2 = L_{g_1} \cdot R_{g_1}^{-1} \cdot g_2$.

3. $SL(2, \mathbb{Z})$ acts on the set of points in the upper half-plane $\mathcal{H} \equiv \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$, by Möbius transformations:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{az+b}{cz+d} \in \mathcal{H}.$$

4. A common group action is the $\mathbb{Z}_2$ grading of a vector space (this plays a role in supersymmetry).

Decompose a vector space V as the sum of two vector subspaces: $V = V_1 \oplus V_2$

Now define a $\underset{\{1,-1\}}{\underset{\parallel}{\mathbb{Z}_2}}$ action by:

$T_1 = \mathbb{1} \in GL(V)$.

$T_{-1} \equiv \sigma \in GL(V)$ acting on the subspaces as:

$$\sigma \cdot v_1 = v_1 \quad \text{and} \quad \sigma \cdot v_2 = -v_2$$
$$\forall\, v_1 \in V_1, \qquad \forall\, v_2 \in V_2.$$

If V has a scalar product such that V_1 and V_2 are orthogonal σ is evidently self-adjoint as $\sigma^2 = \mathbb{1}$. One has a natural notion of even and odd automorphisms:

Even: $V_1 \to V_1$, $V_2 \to V_2$.
Odd: $V_1 \to V_2$, $V_2 \to V_1$.

1.2. Some Representation Theory.

<u>Def$^{\underline{n}}$:</u> A <u>representation</u> of a group G on a vector space V is a group homomorphism:

$$\pi : G \to GL(V).$$

In other words a representation π is a way to write the group G as matrices acting on a vector space which preserves the group composition law. Many groups are naturally written as matrices e.g. $GL(n), SL(n), SO(n), O(n), U(n), SU(n)$ etc however there may be numerous ways to write the group elements as matrices. In addition not all groups can be represented as matrices e.g. S_∞ (the infinite symmetric group) - try writing out an $\infty \times \infty$ matrix (or $GL(\infty), SL(\infty)$... for that matter)

<u>Def$^{\underline{n}}$:</u> If a representation π is such that $Ker(\pi) = e$ then π is a <u>faithful</u> representation.

N.B. that $Ker(\pi)$ is trivial indicates that π is injective or one-to-one: suppose π was not injective and

$\pi(g_1) = \pi(g_2)$ for $g_1 \neq g_2$ $\quad \therefore \pi(g_2^{-1} \cdot g_1) = \mathbb{1}$.

$$\Rightarrow g_2^{-1} \cdot g_1 \in Ker(\pi)$$

hence the Kernel would be non-trivial.

Def$^{\underline{n}}$ A representation $\pi_1(G) \in GL(V_1)$ is <u>equivalent</u> to a second representation $\pi_2(G) \in GL(V_2)$ if $\exists$ an invertible linear map $T : V_1 \to V_2$ such that:

$$T \cdot \pi_1(g) = \pi_2(g) \cdot T$$

T is called the <u>intertwiner</u> from π_1 to π_2.

Def$^{\underline{n}}$: $W \subset V$ is an <u>invariant subspace</u> of a representation $\pi : G \to GL(V)$ if $\pi(g) W \subset W \quad \forall g \in G$.

(W is also called a subrepresentation).

Def$^{\underline{n}}$: An <u>irreducible representation</u> $\pi : GL \to GL(V)$ contains no non-trivial invariant subspaces in V.

(i.e. no $W \subset V$ s.t. $\pi(g) W \subset W \ \forall g \in G$ except $W = V$ or $W = \{e\}$.)

<u>Schur's Lemma.</u>

Let $\pi_1 : G \to GL(V)$ and $\pi_2 : G \to GL(W)$ be irreducible representations of G and let $T : V \to W$ be an intertwining map from π_1 to π_2:

$$T \cdot \pi_1(g) = \pi_2(g) \cdot T.$$

Then either $T = 0$ or T is an isomorphism of V. Furthermore if T is an isomorphism and V is a finite dimensional vector space over $\mathbb{C}$ then $T = \lambda \cdot \mathbb{1}_V$ for some $\lambda \in \mathbb{C}$.