Mechanics, Relativity and Quantum Mechanics.

The purpose of this course is as an introduction to a postgraduate degree in theoretical physics. It aims to ensure that the entire group begins the MSc with a similar foundation. Hence it is taught intensively for the first two weeks of term and will be assessed by examination in January.

Despite the course title the material in the course is split naturally into two parts:

1. Group theory.
2. Mechanics.

Group theory underpins most research in modern theoretical physics and in terms of the subjects in this course will be used to describe special relativity, quantum mechanics and even classical mechanics as well.

The recommended texts are:

Groups, Representations and Physics by H.F. Jones.

Classical Mechanics by H. Goldstein, C. Poole and J. Safko

Quantum Physics by S. Gasiorowicz.

Part I: Group Theory

The first studies of group theory are mostly credited to the famously dead-at-twenty Evariste Galois who was killed in a duel in 1832. Groups were first used to map solutions of polynomial equations into each other. For example the quadratic equation:

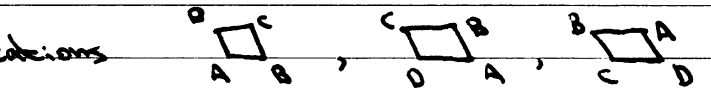
$$y = ax^2 + bx + c$$

is solved when $y=0$ by

$$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right).$$

It has two solutions (+/−) which may be mapped into each other by a $\mathbb{Z}_2$ reflection which swaps the + solution for the − solution. The "$\mathbb{Z}_2$" is the cyclic group of order 2 (also denoted C_n), similarly groups exist which map the roots of general polynomial equation into each other. Groups have a geometrical meaning too. The symmetries which leave unchanged the n-polygons under rotation are also the cyclic groups, $\mathbb{Z}_4$ leaves the square fixed in the plane and consists of all rotations about a fixed point with angle $2\pi/4$.

The cyclic groups are examples of discrete symmetry groups.

Discrete symmetry groups are, in some sense, binary. The action of the discrete group takes a system (e.g. the square in $\mathbb{R}^2$) from one state to another without passing through any of the suspected intervening states. The $\mathbb{Z}_4$ group may act to rotate the square by $\pi/2$ but it does not include in it the rotation of $\pi/4$ for example. One may imagine that the square jumps between the orientations 

On the other hand continuous groups (such as the rotation group in $\mathbb{R}^2$) move the square continuously about the centre of rotation. The rotation is parameterised by a continuous angle variable, θ.

The Norwegian Sophus Lie began the serious study of continuous groups (or Lie groups) in the second half of the 19th century. Rather than thinking about geometry Sophus Lie was interested in whether there were some groups equivalent to Galois groups which mapped solutions of differential equations into each other.* Such groups were identified and classified and named Lie groups. The rotation group ($SO(n)$) is a Lie group.

In the wider context groups may act on more than algebraic equations of 2D geometric shapes and the action of the group

* Very loosely, as each solution to a differential equation is "correct" up to a constant", the solutions have a continuous parameter.

may be encoded in different ways. The study of the ways groups may be represented is aptly named representation theory. It is believed (and proven at the present energies of experiments) that the constituent objects in the universe are invariant under certain symmetries. The standard model of particle physics holds that all known particles are representations of $SU(3)\otimes SU(2)\otimes U(1)$. More simply Einstein's special theory of relativity may be studied as the theory of Lorentz groups.

We will touch most of these topics in this part of the course and we begin with the preliminaries of group theory: what is a group?

1.1. The Basics.

Definition: A group G is a set of elements $\{g, g_2, g_3, \ldots\}$ with a composition law $(\circ)$ which maps $G \times G \to G$ by $(g_1, g_2) \to g_1 \circ g_2$ such that:

ASSOCIATIVE. (i.) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \quad \forall\ g_1, g_2, g_3 \in G$

IDENTITY. (ii.) $\exists\ e \in G$ such that $e \circ g = g \circ e = g \quad \forall\ g \in G$.

INVERSES. (iii.) $\exists\ g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e \quad \forall g \in G$.

Defn.

A group is called commutative or abelian if $g_1 \cdot g_2 = g_2 \cdot g_1$ $\forall\ g_1, g_2 \in G$.

Defn.

The centre $Z(G)$ of a group is:

$$Z(G) \equiv \{ g_1 \in G \mid g_1 \cdot g_2 = g_2 \cdot g_1 \ \forall\ g_2 \in G \}.$$

Defn.

The order $|G|$ of a group G is the number of elements in the set $\{ g_1, g_2, \ldots \}$.

Defn.

For each $g \in G$ the conjugacy class, C_g, is the subset

$$C_g \equiv \{ hgh^{-1} \mid h \in G \} \subset G.$$

Ex. Show that the identity element of a group G is unique.

Sol. For a group G the inverse element g^{-1} of g is unique.* Hence $g^{-1} \cdot g = g \cdot g^{-1} = e$. If there existed a 2nd different identity element, f, s.t. $f \neq e$ then we would have $(g')^{-1} \cdot g = g \cdot (g')^{-1} = f$ and the inverse element would not be unique.

More simply if e and f are two identity elements then:

$$e = e \cdot f = f. \quad \text{(by the definition of the identity).}$$

*Suppose h and k were both inverses to g then: $k = e \cdot k = (h \cdot g) \cdot k = h \cdot (g \cdot k) = h \cdot e = h$. Hence $h = k$.

Examples

1. $G = \{e\}$, the trivial group.

2. $(\mathbb{F}, +)$ where $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

N.B. $\mathbb{Z}$ are the integers $\{\ldots -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{Q}$ are the rationals $\{q/p \mid q, p \in \mathbb{Z} \setminus 0\}$.
$\mathbb{R}$ the real numbers
$\mathbb{C}$ the complex numbers

N.B. under addition the integers $\mathbb{Z}$ are a group as

$a+(b+c) = (a+b)+c \quad \forall\ a,b,c \in \mathbb{Z}$

with identity element 0 and inverse element $-a$ $\forall\ a \in \mathbb{Z}$.

Furthermore it is abelian as $a+b = b+a$.

3. $(\mathbb{F}^\times, \cdot)$ where $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{F}^\times \equiv \mathbb{F} \setminus 0$.

Where $\cdot$ is multiplication.

$E^\times$. Why is $\mathbb{Z}^\times$ not a group under multiplication?

Sol. Consider $a \in \mathbb{Z}^\times$ the inverse under multiplication is $1/a \notin \mathbb{Z}^\times$.

4. $(\mathbb{F}_{>0}, \cdot)$ is an abelian group for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$.

5. $G = \{0, \pm n, \pm 2n, \pm 3n, \ldots\} = n\mathbb{Z}$ where n is a fixed integer in $\mathbb{Z}$. Then $(n\mathbb{Z}, +)$ is a group, an abelian group.

6. $G = \{0, 1, 2, \ldots, n-1\}$ with the composition law $\oplus$

addition modulo n e.g. $a + b = c \bmod n$

7. $G = \{1, -1\}$ under multiplication.

8. $G = \{e, g, g^2, g^3, \ldots g^{n-1}\}$ with $g^k \circ g^l = g^{(k+l) \bmod n}$ and $g^0 \equiv e$.

This is $\mathbb{Z}_n$ the cyclic group of order n.

9. $G = S_n$ the symmetric group or permutation group of n elements.

E.g. S_2 has order $|S_2| = 2!$ and acts on 2 elements:

$((1,2), (2,1))$

The group action is defined element by element:

$g_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ so that $g_1 \circ (1,2) = (2,1)$

and $g_1^2 \circ (1,2) = (1,2)$.

Hence $g_1 = g_1^{-1}$ and $g_1^2 = e$ and $S_2 \equiv \{e, g_1\}$

It is identical to $\mathbb{Z}_2$.

More generally the action of S_n is denoted by a permutation, P such as:

$P = \begin{pmatrix} 1. & 2. & 3. \ldots & n. \\ p_1. & p_2. & p_3. \ldots & p_n \end{pmatrix}$ Where $p_1, \ldots p_n \in \{1, 2, \ldots n\}$.

Which takes $(1, 2, 3, \ldots n)$ to $(p_1, p_2, \ldots p_n)$.

Generally successive permutations do not commute.

For example in S_3 let,

$P = \begin{pmatrix} 1. & 2. & 3. \\ 2. & 3. & 1. \end{pmatrix}$ and $Q = \begin{pmatrix} 1. & 2. & 3. \\ 1. & 3. & 2. \end{pmatrix}$.

Then $Q \cdot P = \begin{pmatrix} 1. & 2. & 3. \\ 1. & 3. & 2. \end{pmatrix}\begin{pmatrix} 1. & 2. & 3. \\ 2. & 3. & 1. \end{pmatrix}$

$= \begin{pmatrix} 1. & 2. & 3. \\ 3. & 2. & 1. \end{pmatrix}$.

while $P \cdot Q = \begin{pmatrix} 1. & 2. & 3. \\ 2. & 3. & 1. \end{pmatrix}\begin{pmatrix} 1. & 2. & 3. \\ 1. & 3. & 2. \end{pmatrix}$

$= \begin{pmatrix} 1. & 2. & 3. \\ 2. & 1. & 3. \end{pmatrix}$.

Hence $P \cdot Q \neq Q \cdot P$ and S_3 is non-abelian.

Additionally one may denote each permutation, P, by its disjoint cycles of elements formed by multiple actions of that permutation, P^n.

E.g. consider P above: $P = \begin{pmatrix} 1. & 2. & 3. \\ 2. & 3. & 1. \end{pmatrix}$

Start with 1 then under P 1 is mapped:

P: $1 \to 2 \to 3 \to 1$.

We may denote this cycle as (1,2,3)

Alternatively Q may be described by two disjoint cycles.

Q: $1 \to 1$ which we denote (1) or ().

Q: $2 \to 3 \to 2$ denoted (2,3).

In all one may write Q as (1)(2,3).

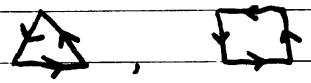
In this notation S_3 is:

$\{ (); (1,2), (1,3), (2,3); (1,2,3), (1,3,2) \}$.

10. The dihedral group D_n, the symmetry group of

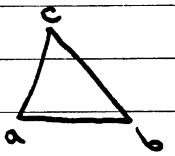
rotations of an n-sided polygon (undirected edges).

N.B. C_n or $\mathbb{Z}_n$ are the rotation symmetries of an n-polygon with directed sides:

 etc

i.e. no reflections are found in $\mathbb{Z}_n$.

While D_n includes the reflections in the plane.

E.g. D_3. Denote 

then $D_3 \equiv \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \right.$

$\left. \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \right\}$

$= \{ (), (a,b), (a,c), (b,c), (a,b,c), (a,c,b) \}$.

Which is identical to S_3.

Ex. What is the order of D_n?

Sol. There are n rotations ($\mathbb{Z}_n$) and one reflection in the line through a vertex and orthogonal to its opposite edge for n odd. (i.e. n reflections for n odd) For even n there is a reflection in the line through opposing vertices and edges, again giving n reflections.

Hence $|D_n| = 2n$.

Ex: Are all dihedral groups D_n identical to the permutation groups S_n?

Sol. No, only $S_3 \cong D_3$, as we have seen.

To see that no other identifications are possible consider the group orders:

$$|S_n| = n! \quad , \quad |D_m| = 2m.$$

And $n! = 2n \Rightarrow n = 3$.

(11.) X a set with $G \equiv \{ f : X \to X \mid f \text{ is a bijection} \}$.

The group action is a composition of maps.

(12.) V, a vector space then $GL(V) \equiv \{ F : V \to V \mid F \text{ is linear and invertible} \}$.

(13.) $(V, +)$ is an abelian group with vector addition.

(14.) $GL(n, \mathbb{F}) \equiv \{ M \in n \times n \text{ matrices} \mid M \text{ is invertible} \}$

The general linear group, with entries in $\mathbb{F} = \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}$.

(15.) $SL(n, \mathbb{F}) \equiv \{ M \in GL(n, \mathbb{F}) \mid \det(M) = 1 \}$.

The special linear group. Or unimodular group.

(16.) $O(n) \equiv \{ M \in GL(n, \mathbb{R}) \mid M^T \cdot M = \mathbb{1}_n \}$.

The orthogonal group.

(17.) $SO(n) \equiv \{ M \in O(n) \mid \det(M) = 1 \}$.

The special orthogonal group.

(18.) $U(n) \equiv \{ M \in GL(n,\mathbb{C}) \mid M^\dagger M = \mathbb{1}_n \}$.

The unitary group.

(19.) $SU(n) \equiv \{ M \in U(n) \mid \det(M) = 1 \}$.

The special unitary group.

(20.) Let $\mathcal{J}_{2n} \equiv \begin{pmatrix} 0. & \mathbb{1}_n. \\ -\mathbb{1}_n. & 0. \end{pmatrix}$

then $Sp(2n) \equiv \{ M \in GL(2n,\mathbb{R}) \mid M^T \mathcal{J}_{2n} M = \mathcal{J}_{2n} \}$.

The symplectic group.

(21.) For $p,q \in \mathbb{Z}_+$ then

$$O(p,q) \equiv \{ M \in GL(p+q,\mathbb{R}) \mid M^T \eta_{p,q} M = \eta_{p,q} \}.$$

Where $\eta_{p,q} = \left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & 0. & & \\ & & & & & \\ 0. & & & -1 & \cdots & -1 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1}_p. & 0. \\ \hline 0. & -\mathbb{1}_q. \end{array}\right)$

(22.) $SL(2,\mathbb{Z}) \equiv \left\{ \begin{pmatrix} a. & b. \\ c. & d. \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$

With matrix multiplication as the composition law.

The modular group.

Remarks.

- (1), (6), (7), (8), (9), (10) satisfy $|G| < \infty$ and are finite groups.
- (14)-(20) are called the classical groups.
- Groups can be represented by giving their multiplication table

e.g. Z_3

	e	g	g^2
e	e	g	g^2
g	g	g^2	e
g^2	g^2	e	g

- Arbitrary combinations of group elements are sometimes called words.
- All $p \in S_n$ can be written as combinations of special elements called transpositions τ_{ij} which swap elements i and j but leave the remainder untouched.

e.g. $p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \tau_{1,2} \circ \tau_{3,1}$

as $p \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 1 & 2 \end{pmatrix}$.

While $\tau_{1,2} \circ \tau_{3,1} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 2 & 3 \end{pmatrix} = \tau_{1,2} \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 2 & 1 \end{pmatrix}$.

If there are N transpositions to replicate p then the $\text{sign}(p) = (-1)^N$.

If we consider only transpositions interchanging consecutive elements ($\tau_{i,i+1}$ and $\tau_{n,1}$)

then the r-cycle:

$$(n_1, n_2, n_3 \dots n_r) = (n_1, n_2)(n_2, n_3) \dots (n_{r-1}, n_r)$$

$$= \tau_{1,2} \tau_{2,3} \dots \tau_{r-1,r}$$

i.e. an r-cycle can be written with $r-1$ transpositions.

The alternating group A_n consists of the even permutations of S_n.

• Cayley's theorem states that every finite group of order n can be considered as a subgroup of S_n.

Def^n. A <u>subgroup</u> H of a group G is a subset of G such that $e \in H$, if $g_1, g_2 \in H$ then $g_1 \cdot g_2 \in H$

$g \in H \Rightarrow g^{-1} \in H$

N.B. the element $\{e\}$ and G itself are trivial subgroups of G by the above definition. If H is not one of these two trivial cases but is still a subgroup then it is called a proper subgroup written $H < G$.

E.g. $S_2 < S_3$ as:

$S_2 = \{ (\,), (1\,2) \}$.

$S_3 = \{ (\,), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2) \}$.

Def^n. Let $H < G$ be a subgroup. The subsets

$$g \cdot H \equiv \{ ghe G \mid h \in H \}$$

are called left-cosets.

$$H \cdot g \equiv \{ h \cdot g \in G \mid h \in H \}$$

are called right-cosets.

N.B. the left-coset gH contains the elements

$\{ gh_1, gh_2, \ldots gh_r \}$

where $r = |H|$ and $\{h_1, h_2 \ldots h_r\}$ are the distinct elements of H. One might suppose that $r < |H|$ if two or more elements of gH were identical but in that case we have:

$$gh_1 = gh_2 \overset{(g^{-1})}{\Rightarrow} h_1 = h_2$$

But h_1 and h_2 were defined to be distinct.

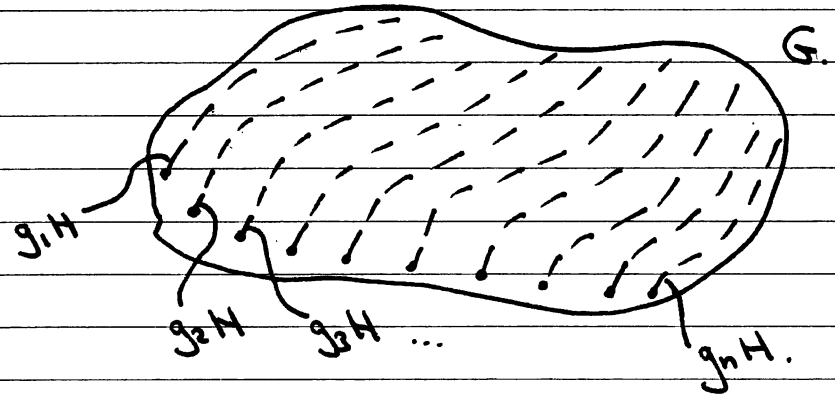
Hence all cosets have the same number of elements.

N.B. 2. Any two left-cosets g_1H and g_2H either coincide or are disjoint. Suppose that $g \in g_1H \cap g_2H$.

so that $g = g_1h_1 = g_2h_2$ for some $h_1, h_2 \in H$.

Then, $g_1H = (gh_1^{-1})H = gH = g(h_2^{-1}H) = g_2H$.

N.B. 3. The left-cosets provide a disjoint partition of G:



Hence $|G| = n|H|$ for some $n \in \mathbb{Z}$.

i.e. we have Lagrange's theorem that the order of any subgroup of G must be a divisor of $|G|$.

A corollary of Lagrange's theorem is that groups of prime order have no proper subgroups (e.g. $\mathbb{Z}_n$ where n is prime).

The same is true for right cosets. The set of cosets itself has n elements (labelled by $g_1, g_2 \dots g_n$) and is denoted G/H.

Defn: $H < G$ is called a **normal subgroup** of G if

$$g \cdot H = H \cdot g$$

and is denoted $H \triangleleft G$.

Note that this is equivalent to saying that

$$g \cdot H \cdot g^{-1} = H$$

or that g commutes with all of H. In particular this is the definition of the centre $Z(G)$ of G, and $Z(G) \triangleleft G$.

Defn: G is called a **simple group** if it has no non-trivial normal subgroups (i.e. besides $\{e\}$ and G itself).

Theorem: If $H \triangleleft G$ is a normal subgroup, then the set of cosets G/H is itself a group with the composition law:

$$(g_1 H) \cdot (g_2 H) = (g_1 g_2 H) \quad \forall \; g_1, g_2 \in H$$

(G/H is then called the factor group or quotient group.)

Proof: (i.) (Associativity)

$$(g_1H)\cdot((g_2H)\cdot(g_3H)) = (g_1H)\cdot(g_2\cdot g_3 H)$$

$$= (g_1\cdot g_2\cdot g_3 H)$$

$$= (g_1\cdot g_2 H)\cdot(g_3H)$$

$$= ((g_1H)\cdot(g_2H))\cdot(g_3H). \quad //$$

(ii.) (Identity) The coset eH acts as the identity:

$$(eH)\cdot(g_1H) = (e\cdot g_1 H) = g_1H.$$

$$(g_1H)\cdot(eH) = (g_1\cdot e H) = g_1H. \quad //$$

(iii.) (Inverse) The inverse of the coset gH is $g^{-1}H$ as:

$$(gH)\cdot(g^{-1}H) = eH = H. \quad //$$

N.B. the group composition law arises as H is a normal subgroup:

$$g_1H\cdot g_2H = g_1\cdot(g_2 H g_2^{-1})\cdot g_2 H$$

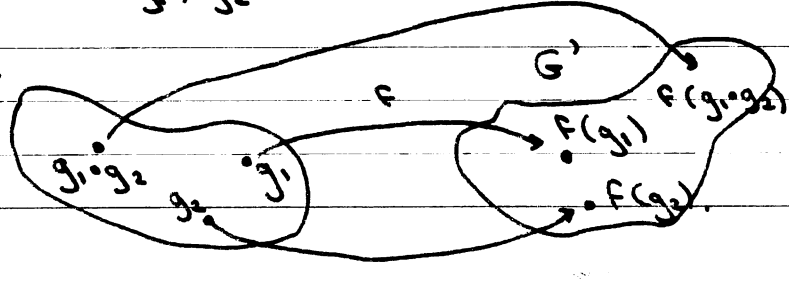
$$= g_1\cdot g_2 H$$

Def^n: A group homomorphism is a map $F: G \to G'$

between two groups $(G,\cdot)$ and $(G',\cdot')$ such that:

$$F(g_1\cdot g_2) = F(g_1)\cdot' F(g_2)$$

$\forall\ g_1, g_2 \in G.$



Def^n. A group **isomorphism** is an invertible group homomorphism. If such an isomorphism exists between G and G' we write $G \cong G'$ and say that G is 'isomorphic to' G'.

Def^n. A group **automorphism** is an isomorphism $F: G \to G$.

N.B. if F is a group homomorphism then in particular:

$$F(e) = e'$$

$$F(g^{-1}) = (F(g))^{-1} \quad \text{as} \quad F(g \circ g^{-1}) = e'$$

$$\| $$

$$F(g) \circ' F(g^{-1}) = (F(g)) \circ' (F(g))^{-1}.$$

Theorem: If $F: G \to G'$ is a group homomorphism then the kernel of F $\ker(F) = \{ g \in G \mid F(g) = e' \}$ is a normal subgroup of G.

Proof: See problem sheet.

A corollary of this is that simple groups G admit only trivial homomorphisms (as $\ker(F) = G$ or $\{e\}$).

Comments.

- $(n\mathbb{Z}, +)$ are abelian hence normal subgroups of $\mathbb{Z}$: $n\mathbb{Z} \trianglelefteq \mathbb{Z}$.
- $(\mathbb{F}_{>0}, \cdot) < (\mathbb{F}^{\times}, \cdot)$.
- Group 6: $(0, 1, 2, \ldots, n-1)$ under addition modulo n is isomorphic to group 7: $(e, g, \ldots g^{n-1})$ with $g^k \circ g^l = g^{(k+l) \bmod n}$ under $F(1) = g$.

- $D_n < S_n$ which is not normal in general.
- $\text{Sign}\ (\rho \in S_n) \to (\{-1, 1\}, \cdot)$ is a group homomorphism.

Consequently the alternating group $A_n \equiv \{\rho \in S_n \mid \text{Sign}(\rho) = 1\}$ is a normal subgroup of S_n since $A_n \equiv \text{Ker}(\text{Sign})$

- $\text{Det}: GL(n, \mathbb{F}) \to (\mathbb{F}^\times, \cdot)$ is a group homomorphism.

$\Rightarrow SL(n, \mathbb{F}) \triangleleft GL(n, \mathbb{F})$ (as $SL(n, \mathbb{F})$ is the Kernel of Det.)

$\Rightarrow SO(n) \triangleleft O(n)$.

$\Rightarrow SU(n) \triangleleft U(n)$.

Also $GL(n, \mathbb{F}) / SL(n, \mathbb{F}) \cong \mathbb{F}^\times$

$O(n) / SO(n) \cong \{1, -1\}$.

$U(n) / SU(n) \cong U(1) \equiv \{z \in \mathbb{C} \mid |z| = 1\}$.

- $Z(SU(2)) = \{\mathbb{1}_2, -\mathbb{1}_2\} \cong \mathbb{Z}_2$ and one can show that the factor group $SU(2) / \mathbb{Z}_2 \cong SO(3)$.

There are a number of simple ways to create new groups from known groups, for example:

(1.) Given a group G, find subgroups H. If these are normal $H \triangleleft G$ then G/H is a group, as shown.

(2.) Given two groups, G and G', find a group homomorphism $f: G \to G'$ such that $\text{Ker}(f) \triangleleft G$ then $G/\text{Ker}(f)$ is a group.