Relativity, Mechanics and Quantum Theory.

Solutions to Problem Sheet 7.

(7.1.) We have:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi.$$

$$\langle \psi | \phi \rangle = \int_{\mathbb{R}^k} d^k q.\ \psi^* \phi.$$

and $$\frac{\partial}{\partial t}\langle \psi|\psi\rangle = \frac{\partial}{\partial t}\int_{\mathbb{R}^k} d^k q\ \psi^*\psi. \equiv \frac{\partial}{\partial t}\int_{\mathbb{R}^k} d^k q\ \rho.$$

i.e. $\rho \equiv \psi^*\psi$.

(i.) $$\therefore \int d^k q \frac{\partial}{\partial t}(\rho) = \int d^k q \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right).$$

N.B. $$-i\hbar \frac{\partial \psi^*}{\partial t} = \hat{H}^*\psi^* = \hat{H}\psi^*.$$

$$\therefore \int d^k q \left(\frac{i}{\hbar}\hat{H}\psi^*\psi + \psi^*\left(\frac{-i}{\hbar}\right)\hat{H}\psi\right).$$

Here $$\hat{H} = -\frac{\hbar^2}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial q_i^2} + \sum_i V(q_i).$$

$$\therefore \int d^k q \frac{\partial}{\partial t}(\rho) = \int d^k q \left(-\frac{i\hbar}{2}\sum_i \frac{1}{m_i}\frac{\partial^2 \psi^*}{\partial q_i^2}\cdot\psi + \psi^*\frac{i\hbar}{2}\sum_i\frac{1}{m_i}\frac{\partial^2\psi}{\partial q_i^2}\right)$$

$$= \int d^k q \left(\cancel{\frac{i\hbar}{2}\sum_i \frac{1}{m_i}\frac{\partial \psi^*}{\partial q_i}\frac{\partial \psi}{\partial q_i}} - \cancel{\frac{i\hbar}{2}\sum_i \frac{1}{m_i}\frac{\partial \psi^*}{\partial q_i}\frac{\partial \psi}{\partial q_i}}\right)$$

$$+ \left[-\frac{i\hbar}{2}\sum_i \frac{1}{m_i}\frac{\partial \psi^*}{\partial q_i}\psi\right]_{\mathbb{R}^k}$$

$$+ \left[+\frac{i\hbar}{2}\sum_i \frac{1}{m_i}\psi^*\frac{\partial \psi}{\partial q_i}\right]_{\mathbb{R}^k}.$$

$$\int d^k q \frac{\partial}{\partial t}(\rho). = \int d^k q \sum_i \frac{\partial}{\partial q_i}\left(-\frac{i\hbar}{2m_i}\left(\frac{\partial \psi^*}{\partial q_i}\psi - \psi^*\frac{\partial \psi}{\partial q_i}\right)\right).$$

$$\therefore \frac{\partial}{\partial t}(\rho) + \sum_i \frac{\partial}{\partial q_i}\underbrace{\left(\frac{i\hbar}{2m_i}\left(\frac{\partial \psi^*}{\partial q_i}\psi - \psi^*\frac{\partial \psi}{\partial q_i}\right)\right)}_{J^i} = 0. \quad //$$

(ii) $\langle\psi|\phi\rangle = \int_{\mathbb{R}^k} d^k q\, \psi^* \phi.$

$$\therefore \frac{d}{dt}\left(\langle\psi|\phi\rangle\right) = \int_{\mathbb{R}^k} d^k q \left(\frac{\partial \psi^*}{\partial t}\phi + \psi^* \frac{\partial \phi}{\partial t}\right).$$

$$= \int_{\mathbb{R}^k} d^k q \left(\frac{i}{\hbar}(\hat{H}\psi^*)\phi + \psi^*\left(\frac{-i}{\hbar}\hat{H}\phi\right)\right).$$

$$= \int_{\mathbb{R}^k} d^k q \left(\frac{i}{\hbar}\left[-\frac{\hbar^2}{2}\sum_i \frac{1}{m_i}\frac{\partial^2 \psi^*}{\partial q_i^2}\cdot\phi + \frac{\hbar^2}{2}\sum_i \frac{1}{m_i}\psi^* \frac{\partial^2 \phi}{\partial q_i^2}\right]\right).$$

$$= \int_{\mathbb{R}^k} d^k q \sum_i \frac{i\hbar}{2m_i}\left(-\frac{\partial^2 \psi^*}{\partial q_i^2}\phi + \psi^* \frac{\partial^2 \phi}{\partial q_i^2}\right).$$

$$= \int_{\mathbb{R}^k} d^k q \sum_i \frac{i\hbar}{2m_i}\left(\cancel{\frac{\partial \psi^*}{\partial q_i}\frac{\partial \phi}{\partial q_i}} - \cancel{\frac{\partial \psi^*}{\partial q_i}\frac{\partial \phi}{\partial q_i}}\right)$$

$$+ \left[\sum_i \frac{i\hbar}{2m_i}\left(-\frac{\partial \psi^*}{\partial q_i}\phi + \psi^* \frac{\partial \phi}{\partial q_i}\right)\right]_{\mathbb{R}^k}.$$

The boundary term vanishes if sufficiently strong conditions are placed on ϕ and ψ. For example if $\phi(\pm\infty) = 0$ and $\psi^*(\pm\infty) = 0$ while $\frac{\partial \psi^*}{\partial q_i}(\pm\infty) \neq \pm\infty$

and $\frac{\partial \phi}{\partial q_i}(\pm\infty) \neq \pm\infty.$

Then the inner product $\langle\psi|\phi\rangle$ is conserved. //

(7.2.) In the Schrödinger picture the states are time-dependent $\psi_S \equiv \psi(q,t)$ while the operators $\hat{A}_S$ are time-independent:

$$\frac{d\hat{A}_S}{dt} = 0$$

In the Heisenberg picture the states are time-independent $\psi_H \equiv \psi(q)$ while the operators $\hat{A}_H$ are time-dependent $\hat{A}_H \equiv A_H(q,t)$.

The states Ψ_S, Ψ_H and operators $\hat{A}_S, A_H$ are related by:

$$|\psi\rangle_H = e^{+i\hat{H}t/\hbar}|\psi(t)\rangle_S$$

(which ~~give~~ comes from a formal solution to the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = e^{-i\frac{t}{\hbar}\hat{H}}|\psi(t)\rangle\Big|_{t=0},$$

$$\therefore \quad \underset{|\psi\rangle_H}{\underset{|||}{|\psi(t)\rangle_{t=0}}} = e^{i\hat{H}t/\hbar}\,\underset{|\psi(t)\rangle_S.}{\underset{|||}{|\psi(t)\rangle}} \;)$$

The operators in the two pictures are related by:

$$\hat{A}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{A}_S\,e^{-i\hat{H}t/\hbar}.$$

The operator matrix elements are unchanged by the picture change:

$${}_H\langle u_n|\hat{A}_H(t)|u_m\rangle_H = {}_S\langle u_n(t)|e^{-i\hat{H}t/\hbar}\,\hat{A}_H(t)\,e^{i\hat{H}t/\hbar}|u_m(t)\rangle_S$$

$$\underset{\left(\hat{A}_H(t)\right)_{nm}}{|||} \qquad = {}_S\langle u_n(t)|\hat{A}_S|u_m(t)\rangle_S.$$

$$\equiv \left(\hat{A}_S\right)_{nm}.$$

for some basis $\{u_n\}$.

The Heisenberg operators evolve in time as:

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i\hat{H}}{\hbar}e^{i\hat{H}t/\hbar}\,\hat{A}_S\,e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar}\,\hat{A}_S\,e^{-i\hat{H}t/\hbar}\left(\frac{-i\hat{H}}{\hbar}\right)$$

$$= \frac{i}{\hbar}\left[\hat{H}, A_H(t)\right]$$

Where we have used $\dfrac{d\hat{A}_S}{dt} = 0.$

(7.3.) $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \dot{x}^2).$

To quantise we treat θ and x as operators.

To find the Hamiltonian we make use of:

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \dot{\theta} \qquad \text{and} \qquad p_x \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}.$$

$$\therefore \quad H = p_i \dot{q}_i - \mathcal{L}$$
$$= \dot{\theta}^2 + \dot{x}^2 - \mathcal{L}$$
$$= \frac{1}{2}(\dot{\theta}^2 + \dot{x}^2)$$
$$\therefore \quad \hat{H} = \frac{1}{2}(\hat{p}_\theta^2 + \hat{p}_x^2).$$

In the position representation $\hat{p}_i \equiv -i\hbar \frac{\partial}{\partial q_i}$

$$\therefore \hat{p}_\theta = (-i\hbar)\frac{\partial}{\partial \theta} \qquad \hat{p}_x = (-i\hbar)\frac{\partial}{\partial x}.$$

$$\therefore \quad \hat{H} = \frac{1}{2}\left(-\hbar^2 \frac{\partial^2}{\partial \theta^2} - \hbar^2 \frac{\partial^2}{\partial x^2}\right)$$
$$= -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}\right).$$

Let $\psi(\theta, x) = \psi(\theta + 2\pi, x)$ be an eigenfunction of $\hat{H}$:

$$\hat{H}\psi = \lambda \psi.$$

As $\hat{H}$ is self-adjoint $\lambda \in \mathbb{R}$.

Since $\psi(\theta, x) = \psi(\theta + 2\pi, x)$ then:

$$\psi(\theta, x) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \phi(x).$$

and,
$$\hat{H}\psi = -\frac{\hbar^2}{2}\left(\sum_{k\in\mathbb{Z}} -k^2 e^{ik\theta}\phi(x) + \sum_{k \in \mathbb{Z}} e^{ik\theta} \cdot \frac{\partial^2 \phi}{\partial x^2}\right) = \lambda \psi.$$

$$\therefore \quad -\frac{\hbar^2}{2}\left(\sum_{k\in\mathbb{Z}} e^{ik\theta}\left(-k^2 \phi(x) + \frac{\partial^2 \phi}{\partial x^2}\right)\right) = \lambda\psi = \lambda \sum_{k\in\mathbb{Z}} e^{ik\theta}\phi(x).$$

$$\Rightarrow \quad -\frac{\hbar^2}{2}\left(-k^2 \phi(x)\right) - \lambda\phi(x) = \frac{\hbar^2}{2}\frac{\partial^2 \phi}{\partial x^2}.$$

$$\Rightarrow \quad \phi(x) = A_k e^{\sqrt{k^2 - 2\lambda/\hbar^2}\, x} + B_k e^{-\sqrt{k^2 - 2\lambda/\hbar^2}\, x}.$$

$$\therefore \ \Psi(x,\theta) = \sum_{k\in\mathbb{Z}} \left(A_k e^{\mu x} + B_k e^{-\mu x}\right) e^{ik\theta}$$

where $\mu = \sqrt{k^2 - 2\lambda/\hbar^2}$.

$$\therefore \quad \mu^2 = k^2 - 2\lambda/\hbar^2.$$

$$\therefore \quad \lambda = \frac{\hbar^2 (k^2 - \mu^2)}{2}. \quad //$$

Where k and μ are determined by the eigenfunction.

(7.4.) We have:

$$\alpha \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)$$

$$\text{and} \quad \alpha^+ \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right).$$

(i.) Using $[\hat{q},\hat{p}] = i\hbar$, we have:

$$[\alpha, \alpha^+] = \frac{m\omega}{2\hbar}\left[\hat{q} + \frac{i}{m\omega}\hat{p}, \hat{q} - \frac{i}{m\omega}\hat{p}\right]$$

$$= \frac{m\omega}{2\hbar}\left(\left[\hat{q}, \frac{-i}{m\omega}\hat{p}\right] + \left[\frac{i}{m\omega}\hat{p}, \hat{q}\right]\right)$$

$$= \frac{m\omega}{2\hbar}\left(\frac{\hbar}{m\omega} + \frac{\hbar}{m\omega}\right)$$

$$= 1. \ //$$

(ii.) $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$$

noting that, $\sqrt{\frac{2\hbar}{m\omega}}\left(\alpha + \alpha^+\right) = 2\hat{q} \quad \Rightarrow \quad \hat{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha + \alpha^+\right)$

$$\sqrt{\frac{2\hbar}{m\omega}}\left(\alpha - \alpha^+\right) = \frac{2i}{m\omega}\hat{p} \quad \text{and} \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\alpha - \alpha^+\right).$$

$$\therefore \quad \hat{H} = \frac{1}{2m}\left(-\frac{\hbar m\omega}{2}(\alpha-\alpha^{+})(\alpha-\alpha^{+})\right) + \frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}(\alpha+\alpha^{+})(\alpha+\alpha^{+})\right)$$

$$= -\frac{\hbar\omega}{4}\left(\cancel{\alpha^2} - \alpha\alpha^{+} - \alpha^{+}\alpha + \cancel{(\alpha^{+})^2}\right) + \frac{\hbar\omega}{4}\left(\cancel{\alpha^2} + \alpha\alpha^{+} + \alpha^{+}\alpha + \cancel{(\alpha^{+})^2}\right)$$

$$= \frac{\hbar\omega}{2}\left(\alpha\alpha^{+} + \alpha^{+}\alpha\right)$$

$$= \frac{\hbar\omega}{2}\left([\alpha, \alpha^{+}] + 2\alpha^{+}\alpha\right)$$

$$= \hbar\omega\left(\tfrac{1}{2} + \alpha^{+}\alpha\right). //$$

(iii.) $$[\hat{H}, \alpha] = \hbar\omega[\alpha^{+}\alpha, \alpha]$$

$$= \hbar\omega[\alpha^{+}, \alpha]\alpha$$

$$= -\hbar\omega\alpha. //$$

$$[\hat{H}, \alpha^{+}] = \hbar\omega[\alpha^{+}\alpha, \alpha^{+}]$$

$$= \hbar\omega\,\alpha^{+}[\alpha, \alpha^{+}].$$

$$= \hbar\omega\,\alpha^{+}. //$$

(iv.) We have, $\hat{H}\,|n\rangle = E_n\,|n\rangle.$

$$\hat{H}\,(\alpha^{+})^k\,|n\rangle = \left(\alpha^{+}\hat{H}(\alpha^{+})^{k-1} + [\hat{H}, \alpha^{+}](\alpha^{+})^{k-1}\right)|n\rangle.$$

$$= \left(\alpha^{+}\hat{H}(\alpha^{+})^{k-1} + \hbar\omega(\alpha^{+})^{k}\right)|n\rangle.$$

$$= \left((\alpha^{+})^k\hat{H} + k\hbar\omega(\alpha^{+})^k\right)|n\rangle.$$

$$= (E_n + k\hbar\omega)\,(\alpha^{+})^k\,|n\rangle. //$$

Similarly,

$$\hat{H}\,(\alpha)^k\,|n\rangle = (E_n - k\hbar\omega)(\alpha)^k\,|n\rangle. //$$

(v.) We note that:

$$\langle n | \frac{\hat{H}}{\hbar w} - \frac{1}{2} | n \rangle = \langle n | n \rangle \left(E_n/_{\hbar w} - \tfrac{1}{2} \right)$$

$$\parallel$$

$$\langle n | \alpha^{+} \alpha | n \rangle \quad = \quad | \alpha | n \rangle |^2 \geqslant 0.$$

$$\therefore \quad E_n \geqslant \tfrac{1}{2} \hbar w.$$

As $\alpha | n \rangle$ is an eigenstate with a lower eigenvalue

i.e. $E_n - \hbar w < E_n$.

and as we have shown there is a lower bound to the set of energy eigenstates then we conclude that not all states $(\alpha)^k | n \rangle$ exist and for some $\hat{k}$ we find

$$\alpha^{\hat{k}} | n \rangle = 0 .$$

The state $\alpha^{\hat{k}-1} | n \rangle \equiv | 0 \rangle$ is the ground state and has energy eigenvalue:

$$\hat{H} | 0 \rangle = \hbar w \left(\tfrac{1}{2} - \alpha^{+} \alpha \right) | 0 \rangle$$

$$= \tfrac{1}{2} \hbar w | 0 \rangle \qquad \text{as} \quad \alpha | 0 \rangle \equiv 0 .$$

Hence the minimum energy is the ground state energy

$$E_0 = \tfrac{1}{2} \hbar w. \; //$$

(vi.) $\langle n | \alpha^{+} \alpha | n \rangle = \langle n | \frac{\hat{H}}{\hbar w} - \frac{1}{2} | n \rangle$

$$= \langle n | n \rangle \left(\frac{E_n}{\hbar w} - \frac{1}{2} \right).$$

Now as $| n \rangle \propto (\alpha^{+})^n | 0 \rangle$ and $E_0 = \frac{1}{2} \hbar w$

the $E_n = E_0 + n \hbar w = (n + \frac{1}{2}) \hbar w$.

$$\therefore \quad \alpha^{+} \alpha | n \rangle = \left((n + \tfrac{1}{2}) - \tfrac{1}{2} \right) | n \rangle = n | n \rangle . \; //$$

(vii) Let us write $|n-1\rangle = \lambda\, \alpha |n\rangle$ for some $\lambda \in \mathbb{C}$.

then
$$\langle n-1|n-1\rangle = 1$$
$$= |\lambda|^2 \langle n|\alpha^+ \alpha|n\rangle.$$
$$= |\lambda|^2\, n \langle n|n\rangle.$$
$$= |\lambda|^2 n$$
$$\Rightarrow |\lambda|^2 = 1/n$$
$$\therefore |\lambda| = 1/\sqrt{n}.$$
$$\therefore \alpha|n\rangle = \sqrt{n}\,|n-1\rangle. \quad //$$

Similarly we have:

(Let $|n+1\rangle = \mu\, \alpha^+ |n\rangle$)

$$1 = \langle n+1|n+1\rangle$$
$$= |\mu|^2 \langle n|\alpha \alpha^+|n\rangle.$$
$$= |\mu|^2 \langle n|[\alpha, \alpha^+] + \alpha^+\alpha|n\rangle$$
$$= |\mu|^2 (1+n) \langle n|n\rangle$$
$$= |\mu|^2 (1+n)$$
$$\Rightarrow |\mu| = 1/\sqrt{1+n}.$$
$$\therefore \alpha^+|n\rangle = \sqrt{1+n}\,|n+1\rangle. \quad //$$