Relativity, Mechanics and Quantum Theory.

Solutions to Problem Sheet 6.

(6.1.) a.). Let $A_R \equiv \int_R dt\, \mathcal{L}(q(t), \dot{q}(t))$.

then under a spacetime transformation:

$$\delta A \equiv A'_{R'} - A_R$$

$$= \underbrace{A'_{R'} - A'_R}_{\text{Temporal transformation.}} + \underbrace{A'_R - A_R}_{\text{Spatial transformation.}}$$

We are concerned, in this problem, with a temporal translation:

$$t' \equiv t + \varepsilon b.$$

The action principle used to derive the equations of motion fixes two points $q(t_1)$ and $q(t_2)$ and extremises the value of the action evaluated over all paths between these points. The path which extremises the action is the dynamical path.

When we consider a temporal translation we do not wish to transform the start and end points of the path, therefore a compensating spatial transformation acts on the coordinates:

$$\underbrace{q(t_1) \rightarrow q(t_1 - \varepsilon b)}_{\substack{\text{Compensating} \\ \text{coordinate} \\ \text{transformation.}}} \underbrace{\longrightarrow}_{\substack{\text{Temporal} \\ \text{translation} \\ t \rightarrow t+\varepsilon b}} q(t_1 + \varepsilon b - \varepsilon b) = q(t_1).$$

Hence $q(t_1)$ and $q(t_2)$ are not shifted, but we have two transformations:

$$t \longrightarrow t + \varepsilon b \qquad \text{and} \qquad q(t) \longrightarrow q(t - \varepsilon b) \simeq q(t) - \varepsilon b \dot{q}(t) + O(\varepsilon^2).$$

From Noether's theorem the conserved charge Q if time translation is a symmetry of the action (i.e. if $\delta A = 0$) is:

$$Q \equiv b\mathcal{L} - b\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i$$
$$= b\mathcal{L} - b\, p_i \dot{q}_i$$
$$= -b(p_i\dot{q}_i - \mathcal{L}).$$

Where $p_i\dot{q}_i - \mathcal{L}$ may be recognised as the energy function.

(6.2.)i.)
$$\mathcal{L} = \frac{m}{2} g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu + e A_\mu \dot{x}^\mu .$$

Let $g_{\mu\nu}$ and A_μ be time-independent:

$$\therefore \quad \frac{d}{dt}(g_{\mu\nu}(x)) = \partial_\rho g_{\mu\nu}\, \dot{x}^\rho = 0.$$
$$\frac{d}{dt}(A_\mu(x)) = \partial_\rho A_\mu \dot{x}^\rho = 0.$$

Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\rho}\right) - \frac{\partial \mathcal{L}}{\partial x^\rho} = 0.$$

$$\therefore \quad \frac{d}{dt}\left(\frac{m}{2} g_{\mu\nu}\, \delta^\mu_\rho \dot{x}^\nu + \frac{m}{2} g_{\mu\nu}\dot{x}^\mu \delta^\nu_\rho + e A_\mu \delta^\mu_\rho\right)$$
$$- \frac{m}{2}(\partial_\rho g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - e(\partial_\rho A_\mu)\dot{x}^\mu = 0.$$

$$\therefore \quad \frac{d}{dt}\left(\frac{m}{2} g_{\rho\nu}\dot{x}^\nu + \frac{m}{2} g_{\mu\rho}\dot{x}^\mu + e A_\rho\right)$$
$$- \frac{m}{2}(\partial_\rho g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - e(\partial_\rho A_\mu)\dot{x}^\mu = 0.$$

$$\therefore \quad m\ddot{x}_\rho + e\cancel{\dot{A}_\rho}^{\,0} - \frac{m}{2}\partial_\rho g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu - e(\partial_\rho A_\mu)\dot{x}^\mu = 0 .$$

The Levi-Civita connection* is defined by:

$$\Gamma^{\kappa}_{\mu\nu} \equiv \frac{1}{2} g^{\kappa\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right).$$

So, $$\Gamma^{\kappa}_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = -\frac{1}{2} g^{\kappa\rho}\, \partial_\rho g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu$$

as $\partial_\mu g_{\rho\nu}\, \dot{x}^\mu = 0.$

Premultiplying the equation of motion by $g^{\kappa\rho}$ gives:

$$m\ddot{x}^\kappa - \frac{m}{2} g^{\kappa\rho}\, \partial_\rho g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu - e\, g^{\kappa\rho}\, \partial_\rho A_\mu\, \dot{x}^\mu = 0.$$

$$\therefore\ m\ddot{x}^\kappa + m\,\Gamma^{\kappa}_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = e\, g^{\kappa\rho}\, \partial_\rho A_\mu\, \dot{x}^\mu. \quad //$$

(ii) The canonical momentum is:

$$p_\rho \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\rho} = \frac{m}{2} g_{\mu\nu}\, \delta^\mu_\rho\, \dot{x}^\nu + \frac{m}{2} g_{\mu\nu}\, \dot{x}^\mu\, \delta^\nu_\rho + e A_\mu\, \delta^\mu_\rho.$$

$$p_\rho = m g_{\rho\nu}\, \dot{x}^\nu + e A_\rho.$$

The Hamiltonian is:

$$p_\rho \dot{x}^\rho - \mathcal{L} = m g_{\rho\nu}\, \dot{x}^\nu \dot{x}^\rho + e A_\rho \dot{x}^\rho - \mathcal{L}.$$

$$= \frac{m}{2} g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu. \quad //$$

Now $$p^\kappa = m\,\delta^\kappa_\nu\, \dot{x}^\nu + e A^\kappa$$

$$\therefore\ \dot{x}^\kappa = (p^\kappa - eA^\kappa)/m.$$

$$\therefore\ H(x^\mu, p_\nu) = \frac{m}{2m^2} g_{\mu\nu}\,(p^\mu - eA^\mu)(p^\nu - eA^\nu).$$

$$= \frac{1}{2m} g_{\mu\nu}\,(p^\mu - eA^\mu)(p^\nu - eA^\nu). \quad //$$

* N.B. the Levi-Civita connection is not part of the course syllabus this year.

(iii.) From question 6.1 we know that the conserved charge if time translation is a symmetry of the motion is

$$p_\kappa \dot{x}^\kappa - \mathcal{L} = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu .$$

Let us show that this is conserved:

$$\frac{d}{dt}\left(\frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right) = \frac{m}{2} g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu + \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \ddot{x}^\nu$$

$$= m g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu .$$

$$= m g_{\kappa\sigma} \ddot{x}^\kappa \dot{x}^\sigma$$

$$= g_{\kappa\sigma} \dot{x}^\sigma \left(-m \Gamma^\kappa_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + e g^{\kappa\rho} \partial_\rho A_\mu \dot{x}^\mu\right)$$

$$= g_{\kappa\sigma} \dot{x}^\sigma \left(+m \dot{x}^\mu \dot{x}^\nu \frac{1}{2} g^{\kappa\rho} \partial_\rho g_{\mu\nu}\right) + e \dot{x}^\sigma \delta^\rho_\sigma \partial_\rho A_\mu \dot{x}^\mu$$

$$= m \dot{x}^\mu \dot{x}^\nu \underbrace{\left(\dot{x}^\rho \partial_\rho g_{\mu\nu}\right)}_{0. \;=\; \left(\frac{d}{dt}(g_{\mu\nu})\right)} + e \underbrace{\left(\dot{x}^\rho \partial_\rho A_\mu\right)}_{0. \;=\; \left(\frac{d}{dt}(A_\mu)\right).} \dot{x}^\mu$$

$$= 0. \quad //$$

(6.3.) $$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2).$$

(a.) $z = x + iy.$

$\bar{z} \equiv z^* = x - iy$

$$\therefore \quad z\bar{z} = (x+iy)(x-iy)$$

$$= x^2 + y^2 .$$

$\dot{z} = \dot{x} + i\dot{y}$

$\dot{\bar{z}} = \dot{x} - i\dot{y}$

$$\therefore \quad \dot{z}\dot{\bar{z}} = \dot{x}^2 + \dot{y}^2 .$$

$$\therefore \quad \mathcal{L} = \frac{m}{2}(\dot{z}\dot{\bar{z}}) - \frac{k}{2} z\bar{z}. \quad //$$

(b.) $z \to z' = e^{iw} z.$

$$\therefore z\bar{z} \to e^{iw} z \, e^{-iw} \bar{z} = z\bar{z}.$$

Similarly (as w is constant):

$$\dot{z} \to \dot{z}' = e^{iw} \dot{z}$$

$$\therefore \dot{z}\dot{\bar{z}} \to \dot{z}\dot{\bar{z}}.$$

$$\therefore \mathcal{L} \to \mathcal{L}' = \frac{m}{2}\left(\dot{z}'\dot{\bar{z}}'\right) - \frac{k}{2}\left(z'\bar{z}'\right)$$
$$= \frac{m}{2}\left(\dot{z}\dot{\bar{z}}\right) - \frac{k}{2}\left(z\bar{z}\right).$$
$$= \mathcal{L}. //.$$

(c.) N.B. under this transformation:

$$z \to z' = e^{iw} z = \left(1 + iw + O(w^2)\right) z$$
$$= z + iwz + O(w^2).$$
$$= z + \delta z + O(w^2).$$

Noether's charge for a spatial transformation is:

$$Q \equiv X^{\mu} \frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}}.$$

Here this becomes:

$$\hat{Q} \equiv \delta z \left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) + \delta \bar{z} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}}\right)$$
$$= (iwz)\left(\frac{m}{2}\dot{\bar{z}}\right) + (-iw\bar{z})\left(\frac{m}{2}\dot{z}\right).$$
$$= \frac{imw}{2}\left(z\dot{\bar{z}} - \bar{z}\dot{z}\right).$$

As w is a constant infinitesimal parameter we may remove it and

$$Q = \frac{im}{2}\left(z\dot{\bar{z}} - \bar{z}\dot{z}\right)$$ is the Noether charge for the symmetry.

(d.) The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{m}{2}\dot{\bar{z}}\right) - \frac{k}{2}\bar{z} = 0.$$

$$\therefore \quad \frac{m}{2}\ddot{\bar{z}} = \frac{k}{2}\bar{z}. \,//$$

$$\therefore \quad m\ddot{\bar{z}} = k\bar{z}. \,// \quad \text{—}(*)$$

$$\text{and} \quad m\ddot{z} = kz. \,// \quad \text{—}(**)$$

Now,

$$\frac{d}{dt}(Q) = \frac{im}{2}\left(\cancel{\dot{z}\dot{\bar{z}}} + z\ddot{\bar{z}} - \cancel{\dot{\bar{z}}\dot{z}} - \bar{z}\ddot{z}\right)$$

Using $(*)$ and $(**)$.

$$= \frac{i}{2}\left(\cancel{z k}\bar{z} - \cancel{\bar{z} k}z\right)$$

$$= 0. \,//.$$