Relativity, Mechanics and Quantum Theory.

Solutions to Problem Sheet 5.

(5.1.) The Lagrangian L is given by:

$$L = T - V$$

Where T is the kinetic energy and V the potential.

Here: $T = \frac{1}{2} m \dot{q}^2$

and $F = -kq = -\frac{d}{dq}(V)$

$$\Rightarrow \quad \frac{\partial V}{\partial q} = kq$$

$$\therefore \quad V = \frac{1}{2} k q^2. \quad \text{up to a constant.}$$

$$\therefore \quad L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2.$$

Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

$$\therefore \quad \frac{d}{dt}\left(m\dot{q}\right) - \left(-kq\right) = 0.$$

$$\therefore \quad m\ddot{q} = -kq.$$

$$\Rightarrow \quad q = A\cos(\omega t) + B\sin(\omega t) \;//$$

Where A, B are constants and $\omega = \sqrt{k/m}$. //

(5.2.) We are given:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu.$$

Where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

If we vary the field A_μ we find the following variation of the Lagrangian:

$$\delta \mathcal{L} = -\frac{1}{4}(\delta F_{\mu\nu})F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}(\delta F^{\mu\nu}) + (\delta A_\mu)J^\mu$$

$$= -\frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} + J^\mu \delta A_\mu.$$

$$= -\frac{1}{2}F^{\mu\nu}\left(\partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)\right) + J^\mu \delta A_\mu.$$

$$= -\frac{1}{2}F^{\mu\nu}\partial_\mu(\delta A_\nu) + \frac{1}{2}F^{\mu\nu}\partial_\nu(\delta A_\mu) + J^\mu \delta A_\mu.$$

$$= -\frac{1}{2}F^{\nu\mu}\partial_\nu(\delta A_\mu) + \frac{1}{2}F^{\mu\nu}\partial_\nu(\delta A_\mu) + J^\mu \delta A_\mu.$$

$$= F^{\mu\nu}\partial_\nu(\delta A_\mu) + J^\mu(\delta A_\mu) \qquad (\text{as } F^{\mu\nu} = -F^{\nu\mu}).$$

Integrate by parts in the action formulation ($\int_R \delta L = \delta A_R$).

$$= \left(-\partial_\nu(F^{\mu\nu}) + J^\mu\right)(\delta A_\mu).$$

(as the total derivative $\partial_\nu(F^{\mu\nu}\delta A_\mu)$ does not contribute to the equation of motion.).

Hence the equation of motion is $-\partial_\nu F^{\mu\nu} + J^\mu = 0$ as δL must vanish for any δA_μ.

Consider the $\mu = 0$ component of the equation of motion:

$$-\partial_\nu F^{0\nu} + J^0 = 0.$$

$$\therefore -\partial_0 \cancel{F^{00}} - \partial_1 F^{01} - \partial_2 F^{02} - \partial_3 F^{03} + J^0 = 0.$$

0.

As $F^{0i} \equiv E_i$ when $i = 1, 2, 3$ then: (and $J^0 = \rho$)

$$-\partial_1 E_1 - \partial_2 E_2 - \partial_3 E_3 + \rho = 0.$$

$$\therefore \quad \underline{\nabla} \cdot \underline{E} = \rho. \;//.$$

When $\mu = i$ ($\equiv \{1, 2, 3\}$) then we have:

$$-\partial_\nu F^{i\nu} + J^i = 0.$$

$$\therefore -\partial_0 F^{i0} - \partial_1 F^{i1} - \partial_2 F^{i2} - \partial_3 F^{i3} + J^i = 0.$$

And $F^{i0} = -E_i$, $F^{ij} = \varepsilon^{ijk} B_k$.

$$\therefore \quad +\partial_0 E_i - \partial_j(\varepsilon^{ijk} B_k) + J_i = 0.$$

$$\therefore \quad -\frac{\partial}{\partial t}(\underline{E}) + \underline{\nabla} \wedge \underline{B} = \underline{J}. \quad //$$

We observe that $\quad \partial_{[\mu} F_{\nu\lambda]} = 0$:

$$\partial_{[\mu} F_{\nu\lambda]} = \frac{1}{3!}\left(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} - \partial_\nu F_{\mu\lambda} - \partial_\mu F_{\lambda\nu} - \partial_\lambda F_{\nu\mu} \right).$$

$$= \frac{1}{3}\left(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} \right).$$

$$= \frac{1}{3}\Big(\partial_\mu(\cancel{\partial_\nu A_\lambda} - \cancel{\partial_\lambda A_\nu}) + \partial_\nu(\cancel{\partial_\lambda A_\mu} - \cancel{\partial_\mu A_\lambda}) + \partial_\lambda(\cancel{\partial_\mu A_\nu} - \cancel{\partial_\nu A_\mu}) \Big).$$

$$= 0. \quad //$$

Consider the components with varying number of timelike indices.

$$\partial_{[0} F_{\nu\lambda]} = 0$$

$$\Rightarrow \quad \partial_0 F_{\nu\lambda} + \partial_\nu F_{\lambda 0} + \partial_\lambda F_{0\nu} = 0.$$ (where ν, λ are spacelike indices)

$$\text{i.e.} \quad \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0.$$

$$\therefore \quad \partial_0(\varepsilon^{ijk} B_k) + \partial_i(-F^{j0}) + \partial_j(-F^{0i}) = 0.$$

$$\therefore \quad \partial_0(\varepsilon^{ijk} B_k) + \partial_i(E_j) + \partial_j(-E_i) = 0.$$

$$\therefore \quad \varepsilon_{ijm}\partial_0(\varepsilon^{ijk} B_k) + \varepsilon_{ijm}(\partial_i E_j - \partial_j E_i) = 0.$$

Now as ε_{ijk} is antisymmetric and $\varepsilon_{123} = +1 = \varepsilon_{231} = \varepsilon_{312}$
$\varepsilon_{213} = -1 = \varepsilon_{132} = \varepsilon_{321}$.

then
$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}.$$

(n.b $i,j,k,l,m,n \in \{1,2,3\}$)

$$\therefore \quad \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

$$\therefore \quad \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}.$$ (also, not needed here, $\varepsilon_{ijk}\varepsilon_{ijk} = 3! = 6$)

So $\quad \varepsilon_{ijm}\, \varepsilon^{ijk}\, \partial_0 B_k + \varepsilon_{ijm}(\partial_i E_j - \partial_j E_i) = 0.$

$$\Rightarrow \quad 2\delta^k_m\, \partial_0 B_k + \varepsilon_{ijm}\, \partial_i E_j - \varepsilon_{jim}\, \partial_i E_j = 0.$$

$$\therefore \quad \cancel{2}\, \partial_0 B_m + \cancel{2}\, \varepsilon_{ijm} \partial_i E_j = 0.$$

$$\therefore \quad \frac{\partial}{\partial t} \underline{B} + \underline{\nabla} \wedge \underline{E} = \underline{0}. \quad //$$

2 timelike indices:

$$\partial_{[0} F_{0\mu]} \equiv \partial_{[0} F_{0i]} = 0$$

$$i = 1, 2, 3$$

$$\Rightarrow \quad \partial_0 F_{0i} + \partial_0 F_{i0} + \partial_i \underbrace{\cancel{F_{00}}}_{0} = 0$$

as $F_{0i} = -F_{i0}$.

$$\Rightarrow \quad 0 = 0. \ //$$

3 timelike indices $\quad \partial_{[0} F_{00]} = 0 \quad \Rightarrow \quad 0 = 0. \ //.$

0 timelike indices:

$$\partial_{[i} F_{jk]} = 0$$

$$\therefore \quad \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$

$$\therefore \quad \partial_i(\varepsilon_{jkl} B_l) + \partial_j(\varepsilon_{kil} B_l) + \partial_k(\varepsilon_{ijl} B_l) = 0.$$

$$\therefore \quad \varepsilon_{ijk}\left[(\partial_i \varepsilon_{jkl} B_l) + \partial_j(\varepsilon_{kil} B_l) + \partial_k(\varepsilon_{ijl} B_l)\right] = 0.$$

$$\therefore \quad \partial_i(2\delta_{il} B_l) + \partial_j(2\delta_{jl} B_l) + \partial_k(2\delta_{kl} B_l) = 0.$$

$$\Rightarrow \quad 6\, \partial_i B_i = 0.$$

$$\therefore \quad \underline{\nabla} \cdot \underline{B} = 0. \quad //$$

(5.3.) The action $A_R[q] \equiv \int_R dt\, \mathcal{L}(q, \dot{q})$ has a space-time transformation symmetry under:

$$q_i \to q_i' = q_i + \varepsilon X_i(q), \qquad t \to t' = t + \varepsilon \xi(t).$$

$$\therefore \quad A_{R'}[q + \delta q] - A_R[q] = 0$$

(as the transformation is a symmetry of $A_R[q] \Rightarrow \delta A_R[q] = 0$)

$$\therefore\ 0 = \left(A_{R'}[q'] - A_R[q']\right) + \left(A_R[q'] - A_R[q]\right)$$

From the first pair of terms we have:

$$A_{R'}[q'] - A_R[q'] = \int_{R'} dt' \, \mathcal{L}(q'(t')) - \int_R dt \, \mathcal{L}(q'(t)).$$

N.B. the full parameter dependency is $\mathcal{L}(q'(t), \dot{q}'(t))$ and here the $\dot{q}'(t)$ has been suppressed for notational simplicity.

$$\therefore\ A_{R'}[q'] - A_R[q'] = \int_R \left(dt + \varepsilon \frac{d\xi}{dt} . dt\right)\left(\mathcal{L}(q') + \varepsilon\xi \frac{\mathcal{L}(q')}{t} + O(\varepsilon^2)\right)$$

$$- \int_R dt \, \mathcal{L}(q').$$

$$= \int_R dt \left(\varepsilon \frac{d\xi}{dt} \mathcal{L}(q') + \varepsilon\xi \frac{\mathcal{L}(q')}{t} + O(\varepsilon^2)\right).$$

$$= \int_R dt \left(\varepsilon \frac{d}{dt}\left(\xi \mathcal{L}(q')\right) + O(\varepsilon^2)\right).$$

$\mathcal{L}(q') = \mathcal{L}(q) + \varepsilon X \frac{\partial \mathcal{L}}{\partial q} + \ldots$

$$= \int_R dt \left(\varepsilon \frac{d}{dt}\left(\xi \mathcal{L}(q)\right) + O(\varepsilon^2)\right)$$

From the second pair of terms above we have:

$$A_R[q'] - A_R[q] = \int_R dt \left(\mathcal{L}(q', \dot{q}') - \mathcal{L}(q, \dot{q})\right)$$

$$= \int_R dt \left(\cancel{\mathcal{L}(q, \dot{q})} + \varepsilon X \frac{\partial \mathcal{L}}{\partial q} + \varepsilon \dot{X} \frac{\partial \mathcal{L}}{\partial \dot{q}} + O(\varepsilon^2) - \cancel{\mathcal{L}(q, \dot{q})}\right).$$

$$= \int_R dt \left(\varepsilon X \frac{\partial \mathcal{L}}{\partial q} + \varepsilon \dot{X} \frac{\partial \mathcal{L}}{\partial \dot{q}} + O(\varepsilon^2)\right)$$

By Lagrange's Eqn. $\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0\right) \rightarrow$

$$= \int_R dt \left(\varepsilon X \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \varepsilon \dot{X} \frac{\partial \mathcal{L}}{\partial \dot{q}} + O(\varepsilon^2)\right)$$

$$= \int_R dt \left(\varepsilon \frac{d}{dt}\left(X \frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + O(\varepsilon^2)\right).$$

Altogether we have:

$$0 = \int_R dt \, \varepsilon \left(\frac{d}{dt}\left(\xi \mathcal{L} + X \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right) + O(\varepsilon^2).$$

Hence $Q \equiv \xi \mathcal{L} + X \frac{\partial \mathcal{L}}{\partial \dot{q}}$ is conserved. //