Relativity, Mechanics and Quantum Theory.

Solutions to Problem Sheet 4.

(4.1) If $T_{\mu\nu}$ is an antisymmetric $(0,2)$ tensor then:

$$T'_{\rho\kappa} = \Lambda_\rho{}^\mu \Lambda_\kappa{}^\nu T_{\mu\nu}.$$

$$= -\Lambda_\kappa{}^\nu \Lambda_\rho{}^\mu T_{\nu\mu} \quad \text{as } T_{\mu\nu} = -T_{\nu\mu}.$$

$$= -T'_{\kappa\rho}.$$

Hence the antisymmetric $(0,2)$ tensors form an invariant subspace.

If $T_{\mu\nu}$ is a symmetric $(0,2)$ tensor then:

$$T'_{\rho\kappa} = \Lambda_\rho^\mu \Lambda_\kappa^\nu T_{\mu\nu}$$

$$= \Lambda_\kappa^\nu \Lambda_\rho^\mu T_{\nu\mu}$$

$$= T'_{\kappa\rho}.$$

$\therefore$ the symmetric $(0,2)$ tensors form an invariant subspace.

If $T_{\mu\nu}$ is a symmetric traceless $(0,2)$ tensor then:

$$\eta^{\mu\nu} T_{\mu\nu} = T^\nu{}_\nu = 0.$$

and $$\eta^{\rho\kappa} T'_{\rho\kappa} = \eta^{\rho\kappa} \left(\Lambda_\rho{}^\mu \Lambda_\kappa{}^\nu T_{\mu\nu} \right).$$

$$= \Lambda_\rho{}^\mu \eta^{\rho\kappa} \Lambda_\kappa{}^\nu T_{\mu\nu}$$

$$= \eta^{\mu\nu} T_{\mu\nu}$$

$$= 0. \quad //$$

$\therefore$ traceless symmetric $(0,2)$ tensors form an invariant subspace.

(4.2) $A \equiv \exp(i \underline{\alpha} \cdot \underline{\sigma})$

Let $\underline{\alpha} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ then $\underline{\alpha} \cdot \underline{\sigma} = \begin{pmatrix} \alpha_3 . & \alpha_1 - i\alpha_2 . \\ \alpha_1 + i\alpha_2 . & -\alpha_3 . \end{pmatrix}$

$$\therefore A = \exp(i \alpha \cdot \underline{\sigma})$$

$$= \mathbb{1} + i \underline{\alpha} \cdot \underline{\sigma} - \frac{1}{2!}(\underline{\alpha} \cdot \underline{\sigma})^2 - \frac{i}{3!}(\underline{\alpha} \cdot \underline{\sigma})^3 + \frac{1}{4!}(\underline{\alpha} \cdot \underline{\sigma})^4 + \ldots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} \alpha_3 . & \alpha_1 - i\alpha_2 . \\ \alpha_1 + i\alpha_2 . & -\alpha_3 . \end{pmatrix} - \frac{1}{2!} \begin{pmatrix} (\alpha_3)^2 + (\alpha_1)^2 + (\alpha_2)^2 . & 0 . \\ 0 . & (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 \end{pmatrix}$$

$$- \frac{i\alpha^2}{3!} \begin{pmatrix} \alpha_3 . & \alpha_1 - i\alpha_2 . \\ \alpha_1 + i\alpha_2 . & -\alpha_3 . \end{pmatrix} + \frac{1}{4!} \alpha^4 \underline{\mathbb{1}} + \ldots$$

$$= \mathbb{1}\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \ldots\right) + \frac{i \underline{\alpha} \cdot \underline{\sigma}}{\alpha}\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \ldots\right)$$

$$= \mathbb{1} \cos(\alpha) + \frac{i \underline{\alpha} \cdot \underline{\sigma}}{\alpha} \sin(\alpha)$$

Where $\alpha \equiv \sqrt{\underline{\alpha} \cdot \underline{\alpha}}$.

$$\therefore A = \begin{pmatrix} \cos(\alpha) + \frac{i\alpha_3}{\alpha}\sin(\alpha) . & \frac{i}{\alpha}(\alpha_1 - i\alpha_2)\sin(\alpha) . \\ \frac{i}{\alpha}(\alpha_1 + i\alpha_2)\sin(\alpha) . & \cos(\alpha) - \frac{i\alpha_3}{\alpha}\sin(\alpha) . \end{pmatrix}$$

Now
$$\det A = \cos^2(\alpha) + \left(\frac{\alpha_3}{\alpha}\right)^2 \sin^2(\alpha) + \frac{(\alpha_1^2 + \alpha_2^2)}{\alpha^2} \sin^2(\alpha)$$

$$= \cos^2(\alpha) + \sin^2(\alpha)$$
$$= 1.$$

Let
$$z_1 \equiv \cos(\alpha) + \frac{i\alpha_3}{\alpha}\sin(\alpha)$$
$$z_2 \equiv \frac{i}{\alpha}(\alpha_1 + i\alpha_2)\sin(\alpha).$$

then
$$A = \begin{pmatrix} z_1 . & -z_2^* . \\ z_2 . & z_1^* \end{pmatrix} \in SU(2)$$

Which is the form of a general element of $SU(2)$ as shown in problem sheet 1.

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i. & 0. \\ 0. & i. \end{pmatrix}$$

$$= \begin{pmatrix} 2i. & 0. \\ 0. & -2i \end{pmatrix}$$

$$= 2i\sigma_3. \quad //$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0. & -1. \\ 1. & 0. \end{pmatrix} - \begin{pmatrix} 0. & 1. \\ -1. & 0. \end{pmatrix}$$

$$= \begin{pmatrix} 0. & -2. \\ 2. & 0. \end{pmatrix}$$

$$= -2i\sigma_2. \quad //$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0. & i. \\ i. & 0. \end{pmatrix} - \begin{pmatrix} 0. & -i. \\ -i. & 0. \end{pmatrix}$$

$$= \begin{pmatrix} 0. & 2i. \\ 2i. & 0. \end{pmatrix}$$

$$= 2i\sigma_1. \quad //$$

$$\therefore \quad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \quad //$$

Where ε_{ijk} is the totally antisymmetric Levi-Civita symbol, with

$$\varepsilon_{123} = +1 = \varepsilon_{231} = \varepsilon_{312}.$$

$$\varepsilon_{213} = -1 = \varepsilon_{132} = \varepsilon_{321}.$$

(4.3.)
$$X = x^\mu \sigma_\mu$$
$$= x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3.$$
$$= x^0\begin{pmatrix}1&0\\0&1\end{pmatrix} + x^1\begin{pmatrix}0&1\\1&0\end{pmatrix} + x^2\begin{pmatrix}0&-i\\i&0\end{pmatrix} + x^3\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$
$$= \begin{pmatrix}x^0+x^3. & x^1-ix^2.\\ x^1+ix^2. & x^0-x^3.\end{pmatrix}.$$

The Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is induced by $X' = AXA^+$ where $A \in SL(2,\mathbb{C})$.

First we show that $Tr(X\sigma_\nu) = 2x^\nu$:

$X\sigma_0 = X.$ $\quad\therefore Tr(X\sigma_0) = 2x^0.$

$X\sigma_1 = \begin{pmatrix}x^1-ix^2. & x^0+x^3.\\ x^0-x^3. & x^1+ix^2.\end{pmatrix}.$ $\quad\therefore Tr(X\sigma_1) = 2x^1.$

$X\sigma_2 = \begin{pmatrix}ix^1+x^2. & -ix^0-ix^3.\\ ix^0-ix^3. & -ix^1+x^2.\end{pmatrix}.$ $\quad\therefore Tr(X\sigma_2) = 2x^2.$

$X\sigma_3 = \begin{pmatrix}x^0+x^3. & -x^1+ix^2.\\ x^1+ix^2. & -x^0+x^3.\end{pmatrix}.$ $\quad\therefore Tr(X\sigma_3) = 2x^3.$

$$\therefore \quad x^\mu = \tfrac{1}{2}Tr(X\sigma_\mu).$$
$$= \tfrac{1}{2}Tr(x^\nu\sigma_\nu\sigma_\mu).$$
$$= \tfrac{1}{2}x^\nu Tr(\sigma_\nu\sigma_\mu).$$

Now under a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$.
and $X \to X' = AXA^+$.

$$\therefore \quad x'^\mu = \tfrac{1}{2}Tr(AXA^+\sigma_\mu)$$
$$= \tfrac{1}{2}Tr(Ax^\nu\sigma_\nu A^+\sigma_\mu)$$
$$= x^\nu \tfrac{1}{2}Tr(A\sigma_\nu A^+\sigma_\mu).$$
$$= \Lambda^\mu{}_\nu x^\nu.$$

$$\therefore \quad \Lambda^\mu{}_\nu(A) = \tfrac{1}{2}Tr(A\sigma_\nu A^+\sigma_\mu). \;//$$

So $x''^{\mu} = \Lambda^{\mu}{}_{\nu} x'^{\nu} = \Lambda^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\lambda} x^{\lambda}. = \Lambda^{\mu}{}_{\nu}(B)\, \Lambda^{\nu}{}_{\lambda}(A)\, x^{\lambda}.$

corresponds to:

$$X'' = B X' B^{+} = B A X A^{+} B^{+} = (BA) X (BA)^{+}.$$

$$\therefore \; x''^{\mu} = \tfrac{1}{2} \mathrm{Tr}\left((BA)\, x^{\nu} \sigma_{\nu} (BA)^{+} \sigma_{\mu}\right)$$
$$= x^{\nu}\, \tfrac{1}{2} \mathrm{Tr}\left((BA) \sigma_{\nu} (BA)^{+} \sigma_{\mu}\right).$$
$$= \Lambda^{\mu}{}_{\nu}(BA)\, x^{\nu}.$$

Hence $\Lambda^{\mu}{}_{\nu}(BA) = \Lambda^{\mu}{}_{\kappa}(B)\, \Lambda^{\kappa}{}_{\nu}(A).$ //

The ~~kernel~~ kernel of the homomorphism $\Lambda : SL(2,\mathbb{C}) \to SO(1,3)$

satisfies $\Lambda^{\mu}{}_{\nu}(A)\, x^{\nu} = x^{\mu} = x'^{\mu}$

i.e. $X' = X$ $\therefore$ $A X A^{+} = X.$ $\forall X$

when $X = \mathbb{1}$ we have:

$$A A^{+} = \mathbb{1} \implies A^{+} = A^{-1}.$$

hence $A \in SU(2).$

and $AX = XA.$

$$\therefore \; [A, x^{\mu} \sigma_{\mu}] = 0.$$

$$\implies [A, \sigma_{\mu}] = 0.$$

As $A \in SU(2).$

$A = \begin{pmatrix} u+iv. & -y+ix. \\ y+ix. & u-iv. \end{pmatrix}, (u^2+v^2+x^2+y^2=1)$

$= u\sigma_0 + x\sigma_1 + i(y\sigma_2 + v\sigma_3).$

$\implies A = u\sigma_0.$

with $u^2 = 1$

$\implies A = \pm\sigma_0 = \pm\mathbb{1}.$ //

Hence the map $\Lambda : SL(2,\mathbb{C}) \to SO(1,3)$ is two-to-one.

1.4) It will suffice to consider a single boost $(x\,t)$ and a single rotation (xy) to find the generators of $SO(1,3)$.

Typical Boost:

$$\Lambda_{B_1} = \begin{array}{c} \\ \begin{pmatrix} \cosh(\theta). & -\sinh(\theta). & 0. & 0. \\ -\sinh(\theta) & \cosh(\theta). & 0. & 0. \\ 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 1. \end{pmatrix} \\ \begin{matrix} \uparrow \quad & \quad \uparrow \quad & \quad \uparrow \quad & \quad \uparrow \\ t \quad & \quad x \quad & \quad y \quad & \quad z. \end{matrix} \end{array} \begin{matrix} \leftarrow t \\ \leftarrow x \\ \leftarrow y \\ \leftarrow z \\ \\ \\ \end{matrix} \qquad (xt \text{ boost}).$$

$$\therefore \left(\frac{d\Lambda_{B_1}}{d\theta}\right)_{\theta=0} = \begin{pmatrix} 0. & -1. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \equiv -iB_1.$$

Typical Rotation:

$$\Lambda_{R_1} = \begin{pmatrix} 1. & 0. & 0. & 0. \\ 0. & \cos\theta. & -\sin\theta. & 0. \\ 0. & \sin\theta. & \cos\theta. & 0. \\ 0. & 0. & 0. & 1. \end{pmatrix}$$

$$\therefore \left(\frac{d\Lambda_{R_1}}{d\theta}\right)_{\theta=0} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & -1. & 0. \\ 0. & +1. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \equiv -iR_3.$$

The remaining boost and rotation generators are found similarly to be:

$$-iB_2 = \begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}, \quad -iB_3 = \begin{pmatrix} 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \end{pmatrix}$$

$$-iR_2 = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \end{pmatrix}, \quad -iR_1 = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 1. & 0. \end{pmatrix}.$$

N.B. note that the generators of spatial rotations (which are compact transformations) are skew-symmetric matrices: $R_i^T = -R_i$. $(i = 1,2,3)$

while the generators of boosts (which are non-compact transformations) are symmetric matrices: $B_i^T = B_i$. $(i = 1,2,3)$.

$$\text{So} \quad [R_3, R_2] = -1\left[\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & -1. & 0. \\ 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \end{pmatrix} - \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \end{pmatrix}\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & -1. & 0. \\ 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\right]$$

$$= -\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. \end{pmatrix} + \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & -1. & 0. \end{pmatrix}$$

$$= iR_1.$$

One also finds (by computation):

$$[R_2, R_1] = iR_3. \qquad \text{and} \qquad [R_1, R_3] = +iR_2.$$

$$\Rightarrow \quad [R_i, R_j] = i\,\varepsilon_{ijk} R_k. \;//$$

For the boosts:

$$[B_1, B_2] = -1\left[\begin{pmatrix} 0. & -1. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} - \begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\begin{pmatrix} 0. & -1 & 0. & 0. \\ -1 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\right]$$

$$= -\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} + \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

$$= -iR_3.$$

We also compute:

$$[B_1, B_3] = iR_2. \qquad \text{and} \qquad [B_2, B_3] = -iR_1.$$

$$\Rightarrow \quad [B_i, B_j] = -i\,\varepsilon_{ijk} B_k. \;//$$

For the mixed commutator:

$$[R_1, B_2] = -1\left[\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 1. & 0. \end{pmatrix}\begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} - \begin{pmatrix} 0. & 0. & -1. & 0. \\ 0. & 0. & 0. & 0. \\ -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -1. \\ 0. & 0. & 1. & 0. \end{pmatrix}\right]$$

$$= -\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0 & 0. & 0. & 0. \\ & 0. & 0. & 0. \end{pmatrix} + \begin{pmatrix} 0. & 0. & 0. & 1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

$$= iB_3.$$

as well as:

$$[R_1, B_3] = -iB_2 \qquad \text{and} \qquad [R_2, B_3] = iB_1.$$

$$\Rightarrow \quad [R_i, B_j] = i\,\varepsilon_{ijk} B_k. \;//$$

Defining $X_j^{\pm} \equiv \frac{1}{2}(R_j \pm iB_j)$. we have:

$$[X_i^+, X_j^+] = \frac{1}{4}[R_i + iB_i, R_j + iB_j]$$
$$= \frac{1}{4}\left([R_i, R_j] + i[R_i, B_j] + i[B_i, R_j] - [B_i, B_j]\right)$$

$$= \frac{1}{4}\left(i\varepsilon_{ijk}R_k + i\left(i\varepsilon_{ijk}B_k - i\varepsilon_{jik}B_k\right) + i\varepsilon_{ijk}R_k\right)$$

$$= \frac{1}{2}i\varepsilon_{ijk}\left(R_k + iB_k\right)$$

$$= i\varepsilon_{ijk}X^+_k.$$

$$[X_i^-, X_j^-] = \frac{1}{4}\left[R_i - iB_i, R_j - iB_j\right].$$

$$= \frac{1}{4}\left([R_i,R_j] - i[B_i,R_j] - i[R_i,B_j] - [B_i,B_j]\right)$$

$$= \frac{1}{4}\left(\varepsilon_{ijk}R_k - i\left(i\varepsilon_{ijk}B_k - i\varepsilon_{jik}B_k\right) + i\varepsilon_{ijk}R_k\right).$$

$$= i\varepsilon_{ijk}X^-_k.$$

$$[X_i^+, X_j^-] = \frac{1}{4}[R_i + iB_i, R_j - iB_j]$$

$$= \frac{1}{4}\left([R_i,R_j] - i[R_i,B_j] + i[B_i,R_j] + [B_i,B_j]\right)$$

$$= \frac{1}{4}\left(\cancel{\varepsilon_{ijk}R_k} - i\left(\cancel{i\varepsilon_{ijk}B_k}\right) + i\left(\cancel{-i\varepsilon_{jik}B_k}\right) - \cancel{\varepsilon_{ijk}R_k}\right).$$

$$= 0.$$