Relativity, Mechanics and Quantum Theory.

Solutions to Problem Sheet 3.

(3.1) The affine transformation is:

$$x \to x' = Ax + b.$$

with A a $D \times D$ matrix
b a constant vector in $\mathbb{R}^D$.

A second affine transformation gives:

$$x' \to x'' = B(Ax+b) + c$$
$$= B \cdot A x + Bb + c$$

with B a $D \times D$ matrix
c a constant vector in $\mathbb{R}^D$.

This transformation can be embedded in $D+1$ dimensions by enhancing x to $\begin{pmatrix} x \\ 1 \end{pmatrix}$ a $D+1$ dimensional vector.
The affine transformation is then:

$$\left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c} x \\ \hline 1 \end{array}\right) = \left(\begin{array}{c} Ax+b \\ \hline 1 \end{array}\right).$$

i.e.
$$\begin{pmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + b_1 . \\ A_{21}x_1 + A_{22}x_2 + b_2 . \\ 1 . \end{pmatrix}$$

Evidently the embedding of x into $\mathbb{R}^{D+1}$ is unique, and there is a one-to-one relation between the affine transformation (A, b) and the matrix representation $\left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right)$.

We can check that it gives a group homomomorphism. From above we see that the composition law is:

$$(B, c) \cdot (A, b) = (B \cdot A, Bb + c)$$

While the representation is

$$\pi(A,b) \equiv \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right).$$

$$\pi(B,c) \equiv \left(\begin{array}{c|c} B. & c. \\ \hline 0. & 1. \end{array}\right)$$

$$\therefore \quad \pi(B,c)\cdot\pi(A,b) = \left(\begin{array}{c|c} B. & c. \\ \hline 0. & 1 \end{array}\right)\left(\begin{array}{c|c} A. & b. \\ \hline 0. & 1. \end{array}\right)$$

$$= \left(\begin{array}{c|c} B\cdot A. & B\cdot b + c. \\ \hline 0. & 1. \end{array}\right).$$

$$= \pi\left(BA, Bb + c\right)$$

$$= \pi\left((B,c)\cdot(A,b)\right). \quad //$$

(3.2.) Let $\pi(g)$ be a finite-dimensional representation of G where $g \in G$.

$$\therefore \quad \pi(g_1 \circ g_2) = \pi(g_1) \circ' \pi(g_2).$$

$\circ$ is the group composition law.

$\circ'$ is matrix multiplication.

Hence also,

$$\pi^*(g_1 \circ g_2) = \pi^*(g_1) \circ' \pi^*(g_2).$$

Where π^* is the complex-conjugate of π.

N.B. if we had taken the transpose the ordering of the r.h.s. would have changed:

$$\pi^T(g_1 \circ g_2) = \left(\pi(g_1) \circ' \pi(g_2)\right)^T$$

$$= \pi^T(g_2) \circ' \pi^T(g_1). \quad //$$

Hence the ordering of g_1 and g_2 is changed and we don't find a simple new homomorphism unless $g_1 \circ g_2 = g_2 \circ g_1$.

(b.) Let $\pi^*(g)$ and $\pi(g)$ be equivalent irreducible representations of G such that:

$$\pi^*(g) = T^{-1}\pi(g)T.$$

$$\therefore \quad (\pi^*(g))^* = (T^{-1})^*\pi^*(g)T^*$$

$$\pi(g) = (T^{-1})^* T^{-1}\pi(g)TT^*.$$

$$= (TT^*)^{-1}\pi(g)TT^*.$$

Hence by Schur's lemma $TT^* = \lambda \mathbb{1}$. //

(c.) Additionally suppose $\pi(g)$ is a unitary representation

$\Rightarrow \quad \langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \quad \forall g \in G,\ v,w \in V.$

Where $\langle\ ,\ \rangle$ is an inner product.

An inner product on $\mathbb{C}^n$ is $\langle v, w \rangle = v^+ w$

$$\therefore \quad \langle \pi(g)v, \pi(g)w \rangle = (\pi(g)v)^+(\pi(g)w)$$

$$= v^+\pi^+(g)\pi(g)w$$

$$= v^+w$$

Hence $\pi^+\pi = \mathbb{1}$ for a unitary representation.

(i.e. $\pi^+ = \pi^{-1}$ and $(\pi^*)^+ = (\pi^*)^{-1}$)

Now, $(\pi^*(g))^{-1} = (T^{-1}\pi(g)T)^{-1}$

$$= T^{-1}\pi^{-1}(g)T.$$

So $\pi^* = ((\pi^*(g))^{-1})^+$

$$= (T^{-1}\pi^{-1}(g)T)^+$$

$$\therefore \quad \pi^*(g) = T^\dagger (\pi^{-1}(g))^\dagger (T^{-1})^\dagger$$
$$= T^\dagger \pi(g) (T^{-1})^\dagger$$

As $\pi^*(g) = T^{-1}\pi(g)T$ then $\pi(g) = T\pi^*(g)T^{-1}$.

$$\therefore \quad \pi(g) = T(T^\dagger \pi(g)(T^{-1})^\dagger)T^{-1}.$$
$$= TT^\dagger \pi(g)(TT^\dagger)^{-1}.$$

As $\pi(g)$ is irreducible then by Schur's lemma

$$TT^\dagger = \mu \mathbb{1}.$$

Now $\det(TT^\dagger) = (\det T)(\det T^\dagger)$

and $\det T^\dagger = (\det T)^*$ as $\det(T) = \det(T^T)$.

$$\therefore \quad \det(TT^\dagger) = (\det T)(\det T)^* \geqslant 0.$$

Hence $(\det(TT^\dagger)) = \mu^n \geqslant 0. \quad \forall \; n$

Where $n = \dim(V)$

$\therefore$ μ is a positive number and $\sqrt{\mu}$ is a well-defined real number.

Hence redefining $T \to \hat{T} = \frac{1}{\sqrt{\mu}} T$ gives:

$$\hat{T}\hat{T}^\dagger = \mathbb{1}$$

and $\hat{T}$ is unitary.

Originally we had $TT^* = \lambda \mathbb{1}$ $\Rightarrow \hat{T}\hat{T}^* = \frac{\lambda}{\mu}\mathbb{1}$.

$$\therefore \quad \hat{T} = \frac{\lambda}{\mu}(\hat{T}^*)^{-1}$$
$$= \frac{\lambda}{\mu}(\hat{T}^*)^\dagger$$
$$= \lambda/\mu \; \hat{T}^T$$

This gives a rule for taking the transpose, hence:

$$\hat{T} = \frac{\lambda^2}{\mu^2}(\hat{T}^T)^T = \frac{\lambda^2}{\mu^2}\hat{T}.$$

$$\therefore \left(\frac{\lambda}{\mu}\right)^2 = +1. \implies \lambda = \pm\sqrt{\mu^2}. = \pm\mu.$$

$$\text{and} \quad \hat{T} = \pm \hat{T}^T.$$

$$\implies T = \pm T^T. \quad //$$

Hence T is either symmetric or antisymmetric.

(3.3.) Let $\pi_1 : G \to GL(V)$

and $\pi_2 : G \to GL(W)$

be two irreducible representations of G on the finite dimensional complex vector spaces V and W.

And suppose that:

$$\pi_2(g) = T_1^{-1}\,\pi_1(g)\,T_1$$

$$\text{and} \quad \pi_2(g) = T_2^{-1}\,\pi_1(g)\,T_2.$$

Then

$$\begin{aligned} T_2\pi_2(g)T_2^{-1} &= \pi_1(g) \\ &\parallel \\ T_2T_1^{-1}\,&\pi_1(g)T_1T_2^{-1} \\ &\parallel \\ (T_1T_2^{-1})^{-1}\,&\pi_1(g)\,(T_1T_2^{-1}). \end{aligned}$$

By Schur's lemma $\quad T_1T_2^{-1} = \lambda\mathbb{1} \quad \lambda \in \mathbb{C}.$

$$\implies T_1 = \lambda T_2. \quad //$$