Relativity, Mechanics and Quantum Theory.

Solutions to problem sheet 2.

(2.1.) a.) Let us check that $M \equiv \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$ is in $SU(2)$ when $|\alpha|^2 + |\beta|^2 = 1$.

$$MM^\dagger = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \alpha\alpha^* + \beta^*\beta & \alpha\beta^* - \beta^*\alpha \\ \beta\alpha^* - \alpha^*\beta & \beta\beta^* + \alpha^*\alpha \end{pmatrix}$$
$$= \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\beta|^2 + |\alpha|^2 \end{pmatrix}.$$
$$= \mathbb{1}_2. \quad //$$

As $|\alpha|^2 + |\beta|^2 = 1$.

Hence $M \in U(2)$.

$$\det(M) = \alpha\alpha^* - \beta(-\beta^*)$$
$$= |\alpha|^2 + |\beta|^2$$
$$= 1. \quad //$$

$\therefore \; M \in SU(2)$.

The matrix M has three free real components. We can see this by writing:

$$\alpha = x + iy.$$
$$\beta = u + iv.$$

then $M = \begin{pmatrix} x+iy & -(u-iv) \\ u+iv & x-iy \end{pmatrix}$ subject to the constraint $\det(M) = (x+iy)(x-iy) + (u+iv)(u-iv) = 1$.

$$= x^2 + y^2 + u^2 + v^2 = 1.$$

$\therefore$ Only 3 of the parameters (x, y, u, v) are independent.

As $SU(2)$ is 3-dimensional and $M \in SU(2)$ and is parameterised by 3 independent variables then M may be used to represent any element of $SU(2)$.

(b.) Let us check the closure of the group:

Let $M_1, M_2 \in SU(2)$ then,

$$(M_1 \cdot M_2)(M_1 \cdot M_2)^\dagger = M_1 \cdot M_2 \cdot M_2^\dagger \cdot M_1^\dagger$$
$$= M_1 \cdot \mathbb{1}_2 \cdot M_1^\dagger$$
$$= \mathbb{1}_2 . //$$

Also $\det(M_1 \cdot M_2) = \det(M_1)\det(M_2)$
$$= 1 . 1 .$$
$$= 1 . //$$

Hence $M_1 \cdot M_2 \in SU(2)$ and the group action $\Rightarrow$ closure.

The group composition law is matrix multiplication which is associative: $M_1 \cdot (M_2 \cdot M_3) = (M_1 \cdot M_2) \cdot M_3$.

The identity element is $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SU(2)$.

(e.g. set $\alpha = 1$, $\beta = 0$ in $M = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$).

The inverse element to M is $M^\dagger = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix}$ which is of the same form as M but $\alpha \to \alpha^*$ and $\beta \to -\beta$.

Hence $M^\dagger = M^{-1} \in SU(2)$.

$\therefore$ $SU(2)$ is a group.

(c.) Let $N \in Z(SU(2))$ and $M \in SU(2)$.

then $N.M = M.N. \Rightarrow M = N^{-1}.M.N.$

Writing $M = \begin{pmatrix} \alpha. & -\beta^*. \\ \beta. & \alpha^*. \end{pmatrix}$ and $N = \begin{pmatrix} a. & -b^*. \\ b. & a^*. \end{pmatrix}$ $\begin{matrix} \alpha, \beta \\ a, b \end{matrix} \in \mathbb{C}$.

with $|\alpha|^2 + |\beta|^2 = 1$. $|a|^2 + |b|^2 = 1$.

Hence $M = N^{-1}.M.N$

$$\Rightarrow \begin{pmatrix} \alpha. & -\beta^*. \\ \beta. & \alpha^*. \end{pmatrix} = \begin{pmatrix} a^*. & b.^*. \\ -b. & a. \end{pmatrix} \begin{pmatrix} \alpha. & -\beta^*. \\ \beta. & \alpha^*. \end{pmatrix} \begin{pmatrix} a. & -b^*. \\ b. & a^*. \end{pmatrix}$$

$$= \begin{pmatrix} a^*. & b^*. \\ -b. & a. \end{pmatrix} \begin{pmatrix} \alpha a. - \beta^* b. & -\alpha b^* - \beta^* a^*. \\ \beta a + \alpha^* b. & -\beta b^* + \alpha^* a^*. \end{pmatrix}$$

$$= \begin{pmatrix} \alpha. a^* a. - \beta^* a^* b. + \beta a b^* + \alpha^* b^* b. & -\alpha a^* b^* - \beta^* a^* a^* - \beta b^* b^* + \alpha^* b^* a^* \\ -\alpha b a. + \beta^* b.b. + \beta a.a. + \alpha^* a.b. & +\beta^* a^* b + \alpha b^*. b. - \beta a. b^* + \alpha^* a a^* \end{pmatrix}$$

The top-left entry $\Rightarrow$ $a^* a \equiv |a|^2 = 1$. $\Rightarrow a = \pm e^{i\theta}$.

$b^* b \equiv |b|^2 = 0$. $\Rightarrow b = b^* = 0$.

From other entries $\Rightarrow$ $a.a = e^{2i\theta} = 1$ $\Rightarrow a = \pm 1$.

Hence $N = \begin{pmatrix} \pm 1. & 0. \\ 0. & \pm 1. \end{pmatrix}$. i.e. $Z(SU(2)) = \{ \mathbb{1}, -\mathbb{1} \} = \mathbb{Z}_2$. //

(d.) Evidently $Z(SU(2))$ is a normal subgroup if it is a subgroup. As it is identified with $\mathbb{Z}_2$ it is a subgroup hence

$$Z(SU(2)) \triangleleft SU(2).$$

(e.) If $h \equiv \begin{pmatrix} \alpha. & 0. \\ 0. & \alpha^*. \end{pmatrix} \in SU(2)$ then $|\alpha|^2 = 1$ $\Rightarrow \alpha = \pm e^{i\theta}$

$$\therefore \quad h = \pm \begin{pmatrix} e^{i\theta}. & 0. \\ 0. & e^{-i\theta}. \end{pmatrix}.$$

Consider first the elements $h = \begin{pmatrix} e^{i\theta} & 0. \\ 0. & e^{-i\theta}. \end{pmatrix}$

When $\theta = 0$ or π $h = \pm \mathbb{1} . \in Z(SU(2))$.

Hence $g.h.g^{-1} = g.g^{-1}.h = h$ $\forall$ $g \in SU(2)$.

$\therefore$ $C_{h=\mathbb{1}} = \{ \mathbb{1} \}$, $C_{h=-\mathbb{1}} = \{ -\mathbb{1} \}$.

(N.B. congucacy class of h is $C_h \equiv \{ g.h.g^{-1} \mid \forall g \in SU(2) \}$).

For other θ:

(let $g = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$)

$$C_h = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}$$

$$= \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} a^* e^{i\theta} & b^* e^{i\theta} \\ -b e^{-i\theta} & a e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} a.a^*.e^{i\theta} + b^*b.e^{-i\theta} & ab^* e^{i\theta} - b^* a e^{-i\theta} \\ b a^* e^{i\theta} - a^* b e^{-i\theta} & b.b^* e^{i\theta} + a^* a e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta + i(aa^* - bb^*)\sin\theta & (\underbrace{ab^* - b^*a}_{0})\cos\theta + i(ab^* + b^*a)\sin\theta \\ (\underbrace{ba^* - a^*b}_{0})\cos\theta + i(ba^* + a^*b)\sin\theta & \cos\theta + i(bb^* - a^*a)\sin\theta \end{pmatrix}$$

Let us write,

$$a = r e^{i\phi} \equiv r\cos\phi + i r \sin\phi$$

$$b = \sqrt{1-r^2}\, e^{i\psi} \equiv \sqrt{1-r^2}\cos\psi + i\sqrt{1-r^2}\sin\psi$$

Then,

$$C_h = \begin{pmatrix} \cos\theta + i(2r^2-1)\sin\theta & 2ir\sqrt{1-r^2}\, e^{i(\phi-\psi)}\sin\theta \\ 2ir\sqrt{1-r^2}\, e^{-i(\phi-\psi)}\sin\theta & \cos\theta - i(2r^2-1)\sin\theta \end{pmatrix}$$

$$\equiv \begin{pmatrix} \hat{\alpha} & -\hat{\beta}^* \\ \hat{\beta} & \hat{\alpha}^* \end{pmatrix}$$

$$\equiv \begin{pmatrix} u+iv & -x+iy \\ x+iy & u-iv \end{pmatrix}$$

N.B. $u + iv = \cos\theta + i((2r^2-1)\sin\theta)$

$x + iy = 2ir\sqrt{1-r^2}\sin\theta(\cos(\phi-\psi) - i\sin(\phi-\psi))$

$= 2r\sqrt{1-r^2}\sin\theta.\sin(\phi-\psi)$

$+ i\,2r\sqrt{1-r^2}\sin\theta.\cos(\phi-\psi)$

Now $u^2 + v^2 + x^2 + y^2 = 1$

$$\therefore \cos^2\theta + (2r^2-1)^2\sin^2\theta + \left(4r^2(1-r^2)\sin^2\theta\sin^2(\phi-\psi)\right)$$
$$+ \left(4r^2(1-r^2)\sin^2\theta.\cos^2(\phi-\psi)\right) = 1.$$

$$\therefore \cos^2\theta + (4r^4 - 4r^2 + 1)\sin^2\theta + 4r^2(1-r^2)\sin^2\theta = 1.$$

$$\therefore 4r^2(r^2-1)\sin^2\theta = 0.$$

$\Rightarrow r = 0,\ r^2 = 1,\ \sin^2\theta = 0.$

More illuminating is the observation that as:

$$u = \cos\theta \quad \text{then} \quad \cos^2\theta + v^2 + x^2 + y^2 = 1.$$

$$\Rightarrow \quad v^2 + x^2 + y^2 = \sin^2\theta.$$

Hence C_h are the 2-spheres of radius $|\sin\theta|$ in the S^3 set of points labelling $SU(2)$. //

Similar arguments apply for $h = -\begin{pmatrix} e^{i\theta}. & 0. \\ 0. & e^{-i\theta}. \end{pmatrix}$.

(2.2.) Let $H \triangleleft G$ then $(g_1 H)\cdot(g_2 H) = (g_1 \circ g_2)H$ for $g_1, g_2 \in G$.

and $g_1 \cdot H = H \cdot g_1$

Let $\left.\begin{matrix} g_1 H \\ g_2 H \end{matrix}\right\} \in \frac{G}{H}$

Then G/H is a group as:

(CLOSURE) $(g_1 H)\cdot(g_2 H) = (g_1 \circ g_2)\cdot H \in \frac{G}{H}$.

(ASSOCIATIVITY)
$$\begin{aligned}(g_1 H)\cdot\big((g_2 H)\cdot(g_3 H)\big) &= (g_1 H)\cdot\big((g_2 \circ g_3)H\big) \\ &= \big(g_1 \circ (g_2 \circ g_3)\big)H \\ &= \big((g_1 \circ g_2)\circ g_3\big)H \\ &= \big((g_1 \circ g_2)H\big)\cdot(g_3 H) \\ &= \big((g_1 H)\cdot(g_2 H)\big)\cdot(g_3 H). //\end{aligned}$$

(IDENTITY) Let e be the identity element in G then eH is the identity element in $\frac{G}{H}$:

$$(eH)\cdot(g_1 H) = (e \circ g_1)H = g_1 H. \quad \forall\, g_1 \in G.$$

(INVERSE) $g_1^{-1} \cdot H$ is the inverse element to $g_1 \cdot H$ as:

$$(g_1^{-1} \cdot H) \cdot (g_1 \cdot H) = (g_1^{-1} \cdot g_1 \cdot H) = (e \cdot H).$$

Therefore G/H is a group.

(2.3.) (This question appeared as stated in the 2008 exam.)

As $\mathbb{Z}_{p_1}$ and $\mathbb{Z}_{p_2}$ are prime order groups Lagrange's theorem states that they both have no non-trival subgroups.

Let $F : \mathbb{Z}_{p_1} \to \mathbb{Z}_{p_2}$ be a homomorphism from $\mathbb{Z}_{p_1}$ to $\mathbb{Z}_{p_2}$.

As $\mathrm{Ker}(F) \triangleleft \mathbb{Z}_{p_1}$ it must be trivial i.e.

$$\mathrm{Ker}(F) = e \quad \text{or} \quad \mathrm{Ker}(F) = \mathbb{Z}_{p_1}.$$

If $\mathrm{Ker}(F) = \mathbb{Z}_{p_1}$ then F is trivial (i.e. $F(a) = e' \ \forall a \in \mathbb{Z}_{p_1}$)

where $e' \in \mathbb{Z}_{p_2}$ is the Identity element in $\mathbb{Z}_{p_2}$.

$\therefore$ If $\mathrm{Ker}(F) = e$ then F is injective.

Now let $a \in \mathbb{Z}_1$ and $e \neq a$ then:

$$F(\underbrace{a +_1 a +_1 a +_1 \dots +_1 a}_{p_2 \text{ terms}}) = F(a) +_2 F(a) +_2 \dots +_2 F(a).$$

$$= 0 \text{ or } e' \text{ in } \mathbb{Z}_{p_2}.$$

Where we note that in $\mathbb{Z}_{p_1}$ $p_2 \cdot a \neq 0$ or e in $\mathbb{Z}_{p_1}$.

Hence the Kernel of the map F is non-trivial (not just e)

$\Rightarrow$ $\mathrm{Ker}(F) = \mathbb{Z}_{p_1}$ and F is trivial.

(2.4.) Let $g_1, g_2 \in G$ then

(a.) The conjugacy class $C_g \equiv \{ \hat{g} \circ g \circ \hat{g}^{-1} \mid \forall\, \hat{g} \in G \}$.

(b.) Consider C_{g_1} and C_{g_2} and suppose $g \in C_{g_1} \cap C_{g_2}$

$$\Rightarrow \quad g = h_1 \circ g_1 \circ h_1^{-1} = h_2 \circ g_2 \circ h_2^{-1}$$

For $h_1, h_2 \in G$.

$$\therefore \quad g_1 = h_1^{-1} \circ h_2 \circ g_2 \circ h_2^{-1} \circ h_1$$
$$\equiv h_3 \circ g_2 \circ h_3^{-1} \in C_{g_2}.$$

$h_3 = h_1^{-1} \circ h_2 \in G$.

$$\therefore \quad C_{g_1} = C_{g_2} \text{ if } C_{g_1} \cap C_{g_2} \neq \emptyset.$$

(c.) The conjugacy class with only one element is that of the identity element e:

$$C_e = \{ g \circ e \circ g^{-1} = e \mid \forall\, g \in G \} = \{e\}.$$

Suppose $g \neq e$ then at least g

C_g

$$e \circ g \circ e^{-1} = g \qquad \text{or} \qquad g \circ g \circ g^{-1} = g \circ e = g.$$

For C_g to form a subgroup we need an identity element, e, in C_g.

Suppose $\exists\, h \in G$ s.t. $h \circ g \circ h^{-1} = e$.

$$\Rightarrow \quad g = h^{-1} \circ e \circ h$$
$$= h^{-1} \circ h$$
$$= e.$$

$\Rightarrow C_g = C_e$ and is of order one.//