Relativity, Mechanics and Quantum Theory.

Solutions to problem sheet 1.

(1.1)

(a) D_3 multiplication table:

	e.	a.	b.	ab.	ba.	b^2.
e.	e.	a.	b.	ab.	ba.	b^2.
a.	a.	e.	a.b	b.	b^2.	ba.
b.	b.	b.a.	b^2.	a.	ab.	e.
ab.	ab.	b^2.	ba.	e.	b.	a.
ba.	ba.	b.	a.	b^2.	e.	ab.
b^2.	b^2.	ab.	e.	ba.	a.	b.

N.B. $(ab)^2 = e$, $a^2 = e$, $b^3 = e$ $\Rightarrow$ $\underset{\overset{\|}{a}}{a(abab)} = a^2 bab = bab.$

and $b^2(a)a = b^2(bab)a$

$\Rightarrow$ $b^2 = aba.$

$\Rightarrow$ $ab^2 = ba.$

& $b^2 a = ab.$

Since $a^2 = b^3 = e$ we may deduce that a is a reflection of the triangle in a symmetry axis while b is a rotation of either $2\pi/3$ or $4\pi/3$.

With this in mind a sensible choice of basis elements of D_3 would be $\{\underbrace{e, b, b^2}_{\text{"Rotations"}}, \underbrace{a, ab, ab^2}_{\text{"Reflections"}}\}$.

Above we used ba instead of ab^2 but recall that they are identical as $ab^2 = ba$.

(b.) From the multiplication table of D_3 we can read off:

Under e: $e \to e$

a: $e \to a \to e$.

b: $e \to b \to b^2 \to e$.

ab: $e \to ab \to e$.

ba: $e \to ba \to e$.

b^2: $e \to b^2 \to b \to e$.

Any element of D_3 can be used as a starting point not just e.

(c.) In terms of permutations:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

satisfy the defining relations of D_3.

(d.) In cyclic notation we have

$$e = (\ )$$
$$a = (1, 2)$$
$$b = (1, 2, 3).$$

The six disjoint cycles of part (b.) are:

$$\{\underset{e.}{(\ )}, \underset{a.}{(1,2)}, \underset{b.}{(1,2,3)}, \underset{ab.}{(1,3)}, \underset{ba}{(2,3)}, \underset{b^2}{(1,3,2)}\}.$$

Which is identical to S_3.

(1.2.) To show that $H = \text{Ker}(f)$ is a subgroup of G (where f is a homomorphism $f: G \to G'$) it is sufficient to demonstrate that H is closed.

If $h_1, h_2 \in H$ then
$$f(h_1 \circ h_2) = f(h_1) \circ' f(h_2)$$
$$= e' \circ e'$$
$$= e'.$$

Hence $h_1 \circ h_2 \in H$.

It may be helpful to convince yourself that the inverse element of $f(h)$ is $(f(h))^{-1}$ as:

$$f(e) = f(h \circ h^{-1}) = \underset{\underset{f(h^{-1}).}{\parallel}}{f(h) \circ' f(h^{-1})} = e' \quad \text{as } e \in H.$$

$$\Rightarrow \quad h^{-1} \in H$$

(N.B.
$\underset{\underset{f(g).}{\parallel}}{f(e \circ g)} = f(e) \circ' f(g) \quad \forall g \in G$
$\therefore f(e) = e'$
$\&\ e \in H.$)

And hence $f(h^{-1}) = (f(h))^{-1}$. //

To show that H is a normal subgroup we must check that:

$$gH = Hg$$

or equivalently that $gHg^{-1} = H$.

Now
$$f(g \circ h \circ g^{-1}) = f(g) \circ' f(h) \circ' f(g^{-1})$$
$$= f(g) \circ' e' \circ' f(g^{-1})$$
$$= f(g) \circ' f(g^{-1}).$$
$$= f(g \circ g^{-1})$$
$$= f(e)$$
$$= e'. \qquad \forall\, h \in H,\ g, g^{-1} \in G.$$

$\therefore \ g \circ H \circ g^{-1} = H.$ //

$\therefore \ \text{Ker}(f) \triangleleft G.$ //

(1.3.) We have:

$$a^2 = b^2 = c^2 = e$$
$$ab = c$$
$$bc = a$$
$$ac = b.$$

(a.) Now,

$$a.b = b^2 a.b. = b^2 c = b.a.$$

,

$$a.c = b^2.a.c = (b.a).(b.c) = (a.b).a = c.a.$$

and

$$b.c = a^2.b.c = (a.b).(a.c) = c.b.$$

Hence V_4 is abelian.

(b.) Write $\mathbb{Z}_2$ as $\{1,-1\}$ with the group action being multiplication.

The elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are

$$(1,1) \equiv e'$$
$$(1,-1) \equiv a'$$
$$(-1,1) \equiv b'$$
$$(-1,-1). \equiv c'.$$

The multiplication law for $\mathbb{Z}_2 \times \mathbb{Z}_2$ is:

$$(a,b) \circ' (\hat{a},\hat{b}) = (a \circ \hat{a}, b \circ \hat{b})$$

Where $\circ$ is multiplication.

$$\therefore (e')^2 = (1,1) \circ' (1,1) = (1,1) = e'.$$
$$(a')^2 = (1,-1) \circ' (1,-1) = (1,1) = e'.$$
$$b'^2 = e'.$$
$$c'^2 = e'.$$

$$a' \circ' b' = (1,-1) \circ' (-1,1) = (-1,-1) = c'.$$
$$a' \circ' c' = (1,-1) \circ' (-1,-1) = (-1,1) = b'.$$
$$b' \circ' c' = (-1,1) \circ' (-1,-1) = (1,-1) = a'.$$

Hence $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$ //

As $A_n \trianglelefteq S_n$ Lagrange's theorem tells us that

$$m|A_n| = |S_n| \quad \text{for some} \quad m \in \mathbb{Z}^+$$

Now $|S_n| = n!$

$$\therefore \quad |A_n| = n!/m.$$

We may directly compute m for a small alternating group. Consider $S_2 \equiv \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$.

$$\text{Sign} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = +1$$

$$\text{Sign} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -1.$$

$$\therefore \quad |A_2| = 1$$

Hence,

$$|A_2| = 2!/m = 1$$

$$\Rightarrow \quad m = 2. //$$

$$\therefore \quad |A_n| = n!/2.$$

In other words half of the permutations of S_n are odd $(\text{Sign}(P) = -1)$ and half are even $(\text{Sign}(P) = +1)$.

(1.4.) $\text{Sign}(P) \equiv (-1)^N$

Where $P \in S_n$ and N is the number of direct transpositions needed to reproduce the permutation P.

N.B. The number of transpositions used to write P is always of the form $N \pm 2m$ (as P may be decomposed in more than one way into transposition maps)

Hence for a given P

$$\text{Sign}(P) = (-1)^{(N \pm 2m)} = (-1)^N (-1)^{\pm 2m} = (-1)^N (1)^{\pm m} = (-1)^N$$

And $\text{Sign}(P)$ is well-defined.

$\text{Sign}(P)$ is a group homomorphism:

$$\text{Sign}(P_1 \circ P_2) = \text{Sign}(P_1) \circ' \text{Sign}(P_2)$$

$$(-1)^{(N_1+N_2)} \;\; \text{III} \;\; = (-1)^{N_1} \circ' (-1)^{N_2} = (-1)^{(N_1+N_2)}$$

($\Rightarrow$ $\circ'$ is multiplication)

Hence $\text{Sign} : S_n \longrightarrow (\{1, -1\}, \times)$ is a group homomorphism.

The alternating group A_n are those $P \in S_n$ satisfying $\text{Sign}(P) = +1$. This is identical to the Kernal of Sign:

$$A_n = \text{Ker}(\text{Sign}).$$

Hence by Q.1.2. A_n is a normal subgroup of S_n:

$$A_n \triangleleft S_n.$$

Hence $\dfrac{S_n}{A_n} \cong \mathbb{Z}_2$ as a quotient group.

$\therefore \quad |S_n| = |A_n||\mathbb{Z}_2|$

and as $|S_n| = n!$, $|\mathbb{Z}_2| = 2$ then

$$|A_n| = n!/2. \quad //$$