1. [2000 STEP I question 5]

   Arthur and Bertha stand at a point O on an inclined plane. The steepest line in the plane through O makes an angle θ with the horizontal. Arthur walks uphill at a steady pace in a straight line which makes an angle α with the steepest line. Bertha walks uphill at the same speed in a straight line which makes an angle β with the steepest line (and is on the same side of the steepest line as Arthur). Show that, when Arthur has walked a distance d, the distance between Arthur and Bertha is $2d\left|\sin\frac{1}{2}(\alpha-\beta)\right|$. Show also that, if $\alpha \neq \beta$ the line joining Arthur and Bertha makes an angle ϕ with the vertical, where

$$\cos\phi = \sin\theta\sin\frac{1}{2}(\alpha+\beta)\,.$$

2. [2004 STEP III question 7] For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

   By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \le t \le 1$ show that $I_{n+1} < \frac{1}{2}I_n$.

   Show also that $I_{n+1} = -\dfrac{1}{n2^n} + I_n$.

   Deduce that $I_n < \dfrac{1}{n2^{n-1}}$.

   Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r2^r} + I_{n+1}$$

   and hence show that

$$\frac{2}{3} < \ln 2 < \frac{17}{24}\,.$$

3. [2004 STEP I question 4]

   Differentiate $\sec t$ with respect to t.

   (i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

   (ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x$.

   (iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x$.

4. [2007 STEP III question 8]

   (a) Find functions $a(x)$ and $b(x)$ such that $u = x$ and $u = e^{-x}$ both satisfy the equation

   $$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a(x)\frac{\mathrm{d}u}{\mathrm{d}x} + b(x)u = 0\,.$$

   For these functions $a(x)$ and $b(x)$ write down the general solutions of the equation.

   Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

   into

   $$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

   and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-e^{-x}}{3(x+e^{-x})}$.

   (b) Find the solution of the equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}\,y = \frac{1}{1-x}$$

   which satisfies $y = 2$ at $x = 0$.

5. [2010 STEP I question 6]

   Show that, if $y = e^x$, then

   $$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

   In order to find other solutions of this differential equation, now let $y = ue^x$, where u is a function of x. By substituting this into $(*)$,show that

   $$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

   By setting $\frac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + Be^x$ satisfies $(*)$, where A and B are any constants.

6. Mechanics [2007 STEP I question 11]

   A smooth, straight, narrow tube of length L is fixed at an agle of 30^o to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motions until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the end of the tunnel. Show that D satisfies the equation

   $$4gD^2 - 2\sqrt{3}(u^2 - Lg)D - 3L(u^2 - gL) = 0$$

   and hence that

   $$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}(u^2 - gL)}\,.$$

   The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

7. Statistics [2004 STEP I question 14]

   Three pirates are sharing out the contents of a treasure chest containing n gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

   Find

   (a) the probability that the first pirate will have some gold coins;

   (b) the probability that the second pirate will have some gold coins;

   (c) the probability that all pirates will have some gold coins.

8. Oxford and Cambridge Schools Examination Board Mathematics and Higher Mathematics Paper II Pure Mathematics July 1965 question 3.

   Prove that, if $t = \tan x$ and $n \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(t^n) = n(t^{n-1} + t^{n+1})\,.$$

   Hence calculate the first five derivatives of $\tan x$ in terms of t and prove that

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots\,.$$

9. [2007 STEP II question 3]

   By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\displaystyle\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}.$

   (a) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x}\mathrm{d}x\,.$

      i. Evaluate I.

      ii. Use the substitution $t = \tan\frac{1}{2}x$ to show that $\displaystyle\int_0^1 \frac{1 - t^2}{1 + 6t^2 + t^4}\mathrm{d}t = \frac{1}{2}I\,.$

   (b) Evaluate $\displaystyle\int_0^1 \frac{1 - t^2}{1 + 14t^2 + t^4}\mathrm{d}t\,.$

10. [2007 STEP I question 9] [Mechanics]

A particle of weight W is placed on a rough plane inclined at an angle θ to the horizontal. The coefficient of friction between the particle and the plane is μ. A horizontal force X acting on the particle is just sufficient to prevent the particle from sliding down the plane; when a horizontal force kX acts on the particle, the particle is about to slide up the plane. Both horizontal forces act in the plane containing the line of greatest slope.

Prove that

$$(k-1)(1+\mu^2)\sin\theta\,\cos\theta = \mu(k+1)$$

and hence that $k > \dfrac{(1+\mu)^2}{(1-\mu)^2}$.

11. [2004 STEP I question 12] [Statistics]

In a certain factory, microchips are made by two machines. Machine A makes a proportion λ of the chips, where $0 < \lambda < 1$, and machine B makes the rest. A proportion p of the chips made by machine A are perfect, and a proportion q of those made by machine B are perfect, where $0 < p < 1$ and $0 < q < 1$. The chips are sorted into two groups: group 1 contains those that are perfect and group 2 contains those that are imperfect.

In a large random sample taken from group 1 it is found that $\frac{2}{5}$ were made by machine A. Show that λ can be estimated as

$$\lambda = \frac{2q}{3p+2q}\,.$$

Subsequently, it is found that the sorting process is faulty: there is a probability $\frac{1}{4}$ that a perfect chip is assigned to group 2 and a probability $\frac{1}{4}$ that an imperfect chip is assigned to group 1. Taking into account this additional information, obtain a new estimate of λ.

12. [2011 STEP I question 8]

   (a) The numbers m and n satisfy

   $$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

   i. Show that $m > n$. Show also that $m < n + 1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n + 1$ unless $-\frac{3}{2} \leq n \leq 0$.

   ii. Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

   (b) Find all integer solutions of the equation $p^3 = q^3 + 2q^2 - 1$.

13. [2011 STEP III question 6] The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{arctanh} t}{t}\,\mathrm{d}t\,,\ U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1 - v^2}\,\mathrm{d}v\,,\ X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x\,.$$

Show, without evaluating any of them, that they are all equal.

14. [2005 STEP I question 9] [Mechanics]

    A non-uniform rod AB has weight W and length $3l$. When the rod is suspended horizontally in equilibrium by vertical strings attached to the ends A and B, the tension in the string attached to A is T.

    When instead the rod is held in equilibrium in a horizontal position by means of a smooth pivot at a distance l from A and a vertical string attached to B, the tension in the string is T. Show that $5T = 2W$.

    When instead the end B of the rod rests on rough horizontal ground and the rod is held in equilibrium at an angle θ to the horizontal by means of a string that is perpendicular to the rod and attached to A, the tension in the string is $\frac{1}{2}T$. Calculate θ and find the smallest value of the coefficient of friction between the rod and the ground that will prevent slipping.

15. [2007 STEP I question 14] [Statistics]

    The discrete random variable X has Poisson distribution with mean λ.

    (a) Sketch the graph of $y = (x+1)e^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

    It is known that $P(X \geq 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

    (b) It is known (instead) that $P(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

    (c) It is known (instead) that $P(X = 1 | X \leq 2) = r$, where r is a number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).