1. [2004 STEP III question 5]

   Show that if $\cos(x - \alpha) = \cos\beta$ then either $\tan x = \tan(\alpha + \beta)$ or $\tan x = \tan(\alpha - \beta)$.

   By choosing suitable values of x, α and β, give an example to show that if $\tan x = \tan(\alpha+\beta)$ then $\cos(x - \alpha)$ need not equal $\cos\beta$.

   Let ω be the angle such that $\tan\omega = \frac{4}{3}$.

   (a) For $0 \le x \le 2\pi$, solve the equatioon

   $$\cos x - 7\sin x = 5$$

   giving both solutions in terms of ω.

   (b) For $0 \le x \le 2\pi$, solve the equation

   $$2\cos x + 11\sin x = 10$$

   showing that one solution is twice the other and giving both in terms of ω.

2. [2005 STEP III question 7]

   Show that if $\displaystyle\int \frac{1}{uf(u)}\,\mathrm{d}u = F(u) + c$, then $\displaystyle\int \frac{m}{xf(x^m)}\,\mathrm{d}x = F(x^m) + c$, where $m \neq 0$.

   Find

   (a) $\displaystyle\int \frac{1}{x^n - x}\,\mathrm{d}x\,;$

   (b) $\displaystyle\int \frac{1}{\sqrt{x^n + x^2}}\,\mathrm{d}x\,.$

3. [2004 STEP II question 9] Mechanics

   The base of a non-uniform solid hemisphere, of mass M, has radius r. The distance of the centre of gravity, G, of the hemisphere from the base is p and from the centre of the base is $\sqrt{p^2+q^2}$. The hemisphere rests in equilibrium with its curved surface on a horizontal plane.

   A particle of mass m, where m is small, is attached to A, the lowest point of the circumference of the base. In the new position of equilibrium, find the angle, α, that the base makes with the horizontal.

   The particle is removed and attached to the point B of the base which is at the other end of the diameter through A. In the new position of equilibrium the base makes an angle β with the horizontal. Show that

$$\tan(\alpha - \beta) = \frac{2mMrp}{M^2(p^2+q^2) - m^2r^2}\,.$$

4. [2004 STEP II question 12] Statistics

   Sketch the graph, for $x \geq 0$, of

$$y = kxe^{-ax^2}\,,$$

   where a and k are positive constants.

   The random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} kxe^{-ax^2} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Show that $k = \dfrac{2a}{1-e^{-a}}$ and find the mode m in terms of a, distinguishing between the cases $a < \frac{1}{2}$ and $a > \frac{1}{2}$.

   Find the median h in terms of a and show that $h > m$ if $a > -\ln\left(2e^{-\frac{1}{2}} - 1\right)$.

   Show that $-\ln\left(2e^{-\frac{1}{2}} - 1\right) > \frac{1}{2}$. Show that, if $a > -\ln\left(2e^{-\frac{1}{2}} - 1\right)$, then

$$P(X > m | X < h) = \frac{2e^{-\frac{1}{2}} - e^{-a} - 1}{1 - e^{-a}}\,.$$