1. [2004 STEP III question 5]

   (a)
   $$\cos x - 7\sin x = \sqrt{50}\cos(x-\alpha)$$
   where $\cos\alpha = \frac{1}{\sqrt{50}}$ and $\sin\alpha = -\frac{7}{\sqrt{50}}$.
   Also since $\cos(x-\alpha) = \frac{1}{\sqrt{2}}$, $\tan(x-\alpha) = \pm 1$.
   Using the first part, it can then be shown that either $\tan x = \frac{4}{3}$ or $\tan x = -\frac{3}{4}$.
   Hence $x = \omega$ or $x = \frac{3\pi}{2} + \omega$.

   (b) Similar arguments give $\tan x = \frac{4}{3}$ or $\tan x = -\frac{24}{7}$.
   The two solutions are ω and 2ω.

2. [2005 STEP III question 7]
   Let $u = x^m$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = mx^{m-1}$ and so
   $\int \frac{m}{xf(x^m)}\,\mathrm{d}x = \int \frac{m}{mx^m f(u)}\,\mathrm{d}u = \int \frac{1}{uf(u)}\,\mathrm{d}u = F(u) + c = F(x^m) + c.$

   (a) Using $f(u) = u - 1$ and $m = n - 1$,
   $$\int \frac{1}{x^n - x}\,\mathrm{d}x = \frac{1}{n-1}\ln\left(\frac{x^{n-1}-1}{x^{n-1}}\right) + c\,;$$

   (b) Using $f(u) = \sqrt{u+1}$ and $m = n - 2$ gives
   $$\int \frac{1}{\sqrt{x^n + x^2}}\,\mathrm{d}x = \frac{1}{n-2}\ln\left|\frac{\sqrt{x^{n-2}+1}-1}{\sqrt{x^{n-2}+1}+1}\right| + c\,.$$

3. [2004 STEP II question 9] Mechanics

   $\tan\alpha = \dfrac{mr + Mq}{Mp}$.

   $\tan\beta = \dfrac{-mr + Mq}{Mp}$.

   (Hence, using formula for $\tan(\alpha - \beta)$,

   $$\tan(\alpha - \beta) = \frac{2mMrp}{M^2(p^2 + q^2) - m^2r^2} \text{ .)}$$

4. [2004 STEP II question 12] Statistics

   The mode m is $\frac{1}{\sqrt{2a}}$ if $a > \frac{1}{2}$ and is 1 when $a < \frac{1}{2}$.

   The median $h = \sqrt{\dfrac{1}{a} \ln \dfrac{2}{1 + e^{-a}}}$.