1. [2000 STEP I question 5]

   The question includes its own solutions in that results to be proved are stated.

   En route to the second result you may find it useful to calculate the heights of Arthur and Bertha above the horizontal.

   These heights are $d\cos\alpha\sin\theta$ and $d\cos\beta\sin\theta$ respectively.

2. [2004 STEP III question 7] The question includes its own solutions in that results to be proved are stated.

3. [2004 STEP I question 4]

   (i) Using the substitution $x = \sec t$ gives

   $$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 t\mathrm{d}t\,.$$

   (ii) $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x = \sec^{-1}(x+2) + c\,.$

   (iii) $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x = \frac{1}{3}\sec^{-1}\left(\frac{x+2}{3}\right) + c\,.$

4. [2007 STEP III question 8]

   (a) $a(x) = \frac{1}{1+x}$ and $b(x) = \frac{-1}{1+x}$.
   General solution $u(x) = Ax + Be^{-x}$.

   (b) Use $y = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$ and

   $$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1-x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1-x}u = 0$$

   which has general solution $Cx + De^x$.
   Solution $y = \frac{1+e^x}{x+e^x}$.

[2010 STEP I question 6]

If $y = e^x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, so that direct substitution shows that (*) is satisfied.

If $y = ue^x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = (u + \frac{\mathrm{d}u}{\mathrm{d}x})e^x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (u + 2\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}^2u}{\mathrm{d}x^2})e^x$. Substituting these expressions into $(*)$ gives

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

Setting $\frac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ gives

$$(x-1)\frac{\mathrm{d}v}{\mathrm{d}x} + (x-2)v = 0$$

which has solution $v = ke^{-x}(x-1)$ where k is an arbitrary constant. Since $v = \frac{\mathrm{d}u}{\mathrm{d}x}$, integration gives $u = -kxe^{-x} + c$ with c a further arbitrary constant.

Finally, recalling that $y = ue^x$ satisfies $(*)$, we obtain $y = (-kxe^{-x}) \times e^x = -kx + ce^x$ and so $y = Ax + Be^x$ satisfies $(*)$, where $A = -k$ and $B = c$ and hence A and B are arbitrary constants.

5. [2007 STEP I question 11]

The speed v of the particle at the moment when it leaves the tube satisfies $v^2 = u^2 - gL$.

After that

$$x = \frac{\sqrt{3}}{2}vt\,, \qquad y = \frac{1}{2}(vt - gt^2)$$

where t is time measured from when the particle leaves the tube, and the origin of coordinates x, y is the top end of the tube.

Then if T is the time when the particle is again level with the bottom of the tube

$$D = \frac{\sqrt{3}}{2}vT \quad \text{and} \quad L + vT - gT^2 = 0\,.$$

Hence

$$4gD^2 - 2\sqrt{3}(u^2 - Lg)D - 3L(u^2 - gL) = 0$$

and so

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}(u^2 - gL)}\,.$$

Finally $R = \sqrt{3}L = \dfrac{2u^2}{\sqrt{3}g}\,.$

6. [2004 STEP I question 14]

   (a) $\frac{n}{n+2}$

   (b) The second pirate will get some gold if the two lead coins do not come in succession. There are $(n+2)!$ different arrangements of the coins. Of these, there are $(n+1)! \times 2!$ where the lead coins are adjacent. Hence the probability that there are two successive lead coins is $\frac{2}{n+2}$ and so the probabailty that the two lead coins do not come in succession is $\frac{n}{n+2}$.

   (c) $\frac{(n-1)(n-2)}{(n+1)(n+2)}$