1. [2007 Oxford Admission Test question 1C] (Multiple Choice)

   The number of solutions x to the equation

   $$7\sin x + 2\cos^2 x = 5\,,$$

   in the range $0 \le x < 2\pi$, is

   (a) 1 (b) 2 (c) 3 (d) 4

2. [2007 Oxford Admission Test question 1F] (Multiple Choice)

   The equation

   $$8^x + 4 = 4^x + 2^{x+2}$$

   has

   (a) no real solutions (b) 1 real solution (c) 2 real solutions (d) 3 real solutions

3. [2007 Oxford Admission Test question 1I] (Multiple Choice)

   Given that a and b are positive and

   $$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1\,,$$

   then the greatest possible value of a is

   (a) $\frac{1}{10}$ (b) 1 (c) $\sqrt{10}$ (d) $10^{\sqrt{2}}$

4. [2007 Oxford Admission Test question 1J] (Multiple Choice)

   The inequality

   $$(n+1) + (n^2+2) + (n^3+3) + \cdots + (n^{10000}+100) > k$$

   is true for all $n \ge 1$. It follows that

   (a) $k < 1300$
   (b) $k^2 < 101$
   (c) $k > 10110000$
   (d) $k < 5150$

5. [2007 Oxford Admission Test question 3]

   Let

$$I(c) = \int_0^1 \left((c-x)^2 + c^2\right) \mathrm{d}x$$

   where c is a real number.

   (a) Sketch $y = (x-1)^2 + 1$ for the values $-1 \leq x \leq 3$ and show on your graph the area represented by the integral $I(1)$.

   (b) Without explicitly calculating $I(c)$, explain why $I(c) > 0$ for any value of c.

   (c) Calculate $I(c)$.

   (d) What is the minimum value of $I(c)$ (as c varies)?

   (e) What is the maximum value of $I(\sin\theta)$ as θ varies?

This week's theme: thinking carefully about calculus

6. Prove from first principles that if $y = \dfrac{1}{x}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x^2}$.

   [If you don't know what this means, consider the curve $y = \dfrac{1}{x}$ and calculate the gradient of the chord joining $(x+h, \frac{1}{x+h})$ and $(x, \frac{1}{x})$ where x and $x+h$ are both positive. What happens as $h \to 0$? Relate this to the gradient of the tangent to the curve at $(x, \frac{1}{x})$.]

7. [2008 Oxford Admission Test question 1A] (Multiple Choice)

   The function $y = 2x^3 - 6x^2 + 5x - 7$ has

   (a) no stationary points
   (b) one stationary point
   (c) two stationary points
   (d) three stationary points

8. [2008 Oxford Admission Test question 1F] (Multiple Choice)

   If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,\mathrm{d}x$$

   by splitting the interval $0 \le x \le 1$ into 10 intervals then an overestimate of the integral is produced. It follows that

   (a) the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\,\mathrm{d}x$
   (b) the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x) - 1)\,\mathrm{d}x$
   (c) the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\,\mathrm{d}x$
   (d) the trapezium rule with 10 intervals underestimates $\int_0^1 (1 - f(x))\,\mathrm{d}x$

9. [2009 Specimen Paper 1 Oxford Admission Test question 1B] (Multiple Choice)
The smallest value of the function
$$f(x) = 2x^3 - 9x^2 + 12x + 3$$
in the range $0 \leq x \leq 2$ is
(a) 1 (b) 3 (c) 5 (d) 7

10. [2007 AEA question 2]
    (a) On the same diagram, sketch $y = x$ and $y = \sqrt{x}$, for $x \geq 0$, and mark clearly the coordinates of the points of intersection of the two graphs.
    (b) With reference to your sketch, explain why there exists a value a of x ($a > 1$) such that
    $$\int_0^a x \,\mathrm{d}x = \int_0^a \sqrt{x} \,\mathrm{d}x\,.$$
    (c) Find the exact value of a.
    (d) Hence, or otherwise, find a non-constant function $f(x)$ and a constant b ($b > 0$) such that
    $$\int_{-b}^{b} f(x) \,\mathrm{d}x = \int_{-b}^{b} \sqrt{[f(x)]} \,\mathrm{d}x\,.$$

11. [2004 STEP I question 2]
The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $\left[\sqrt{24}\right] = 4$ and $[5] = 5$.
    (a) Sketch the graph of $y = \sqrt{[x]}$ and show that
    $$\int_0^a \sqrt{[x]} \,\mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$
    when a is a positive integer.
    (b) Show that
    $$\int_0^a 2^{[x]} \,\mathrm{d}x = 2^a - 1$$
    when a is a positive integer.
    (c) Determine an expression for
    $$\int_0^a 2^{[x]} \,\mathrm{d}x$$
    when a is positive but not an integer.

This week you are asked to get in a group of two, three or four where everyone is from a different school or college, and there is a mixture of gender.

Also, you are asked to take care to justify every statement you make. Discuss together how to write down well-argued answers to each question. (Do this even for multiple choice questions.)

12. Warm up

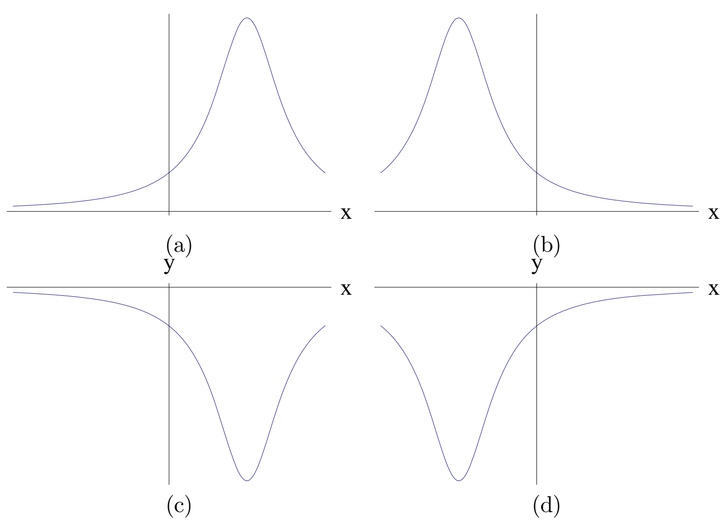
    (a) Evaluate $1 + 4 + 7 + \cdots + 22$.

    (b) What is the remainder when $x^3 + 3x^2 - 4x + 7$ is divided by $x + 2$?

13. [2008 Oxford Admission Test question 1G] (Multiple choice)

    Which of the graphs below is a sketch of

$$y = \frac{1}{4x - x^2 - 5} \ ?$$



14. [2008 Oxford Admission Test question 1D] (Multiple Choice)

    When

    $$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

    is divided by $x - 1$ the remainder is

    (a) 2000 (b) 2500 (c) 3000 (d) 3500

15. [2008 Oxford Admission Test question 1J] (Multiple Choice)

    In the range $0 \le x < 2\pi$ the equation $(3 + \cos x)^2 = 4 - 2\sin^8 x$ has

    (a) 0 solutions (b) 1 solution (c) 2 solutions (d) 3 solutions

16. [2004 STEP I question 5]

    The positive integers can be split into five distinct arithmetic progressions, as shown:

    $$\begin{aligned} A &: 1,\ 6,\ 11,\ 16,\ldots \\ B &: 2,\ 7,\ 12,\ 17,\ldots \\ C &: 3,\ 8,\ 13,\ 18,\ldots \\ D &: 4,\ 9,\ 14,\ 19,\ldots \\ E &: 5, 10,\ 15,\ 20,\ldots \end{aligned}$$

    Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

    Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

    (i) Prove that there are no positive integers x and y such that

    $$x^2 + 5y = 243\,723\,.$$

    (ii) Prove also that there are no positive integers x and y such that

    $$x^4 + 2y^4 = 26\,081\,974\,.$$

17. Warm up

    Given $y = x^2(x-3)^3$, evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ and simplify the result. (Do not multiply out the brackets!)

18. Warm up

    Write down the formulae for $\sin(A+B)$ and $\cos(A+B)$ and use them to prove the formula for $\tan(A+B)$.

19. [2005 STEP 1 question 4]

    (a) Given that $\cos\theta = \frac{3}{5}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$, show that $\sin 2\theta = -\frac{24}{25}$, and evaluate $\cos 3\theta$.

    (b) Prove the identity

    $$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\,.$$

    Hence evaluate θ given that $\tan 3\theta = \frac{11}{2}$ and $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

20. [2004 STEP I question 3]

    (a) Show that $x-3$ is a factor of

    $$x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y\,. \qquad (*)$$

    Express (*) in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

    (b) Factorise $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ into three linear factors.

21. [2009 STEP II question 4]

    The polynomial $p(x)$ is of degree 9 and $p(x) - 1$ is exactly divisible by $(x-1)^5$.

    (a) Find the value of $p(1)$.

    (b) Show that $p'(x)$ is exactly divisible by $(x-1)^4$.

    (c) Given also that $p(x) + 1$ is exactly divisible by $(x+1)^5$, find $p(x)$.

22. [1999 STEP I question 3]

The n positive numbers $x_1, x_2, \ldots, x_n$, where $n \geq 3$, satisfy

$$x_1 = 1 + \frac{1}{x_2}, \quad x_2 = 1 + \frac{1}{x_3}, \quad \ldots, x_{n-1} = 1 + \frac{1}{x_n}, \quad \text{and also} \quad x_n = 1 + \frac{1}{x_1}.$$

Show that

(a) $x_1, x_2, \ldots, x_n > 1$.

(b) $x_1 - x_2 = -\frac{x_2 - x_3}{x_2 x_3}$.

(c) $x_1 = x_2 = \cdots = x_n$.

Hence find the value of x_1.

23. Warm up

Let $f(x) = x + 3$ and $g(p) = \dfrac{p}{2}$.

Evaluate $f^{-1}(z), g^{-1}(q), fg(t), gf(m)$, and verify that $f^{-1}g^{-1}(y) = (gf)^{-1}(y)$.

24. [2011 AEA question 1]

Solve, for $0 \leq \theta \leq 180^o$,

$$\tan(\theta + 35^o) = \cot(\theta - 53^o)\,.$$

25. [2011 AEA question 7]

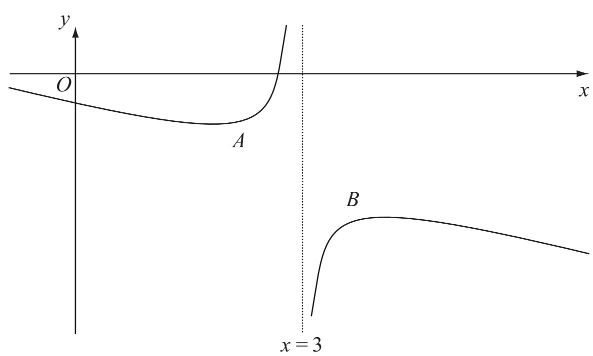


Figure 4

(a) Figure 4 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R},\ x \neq 3$$

The curve has a minimum at the point A, with x-coordinate α, and a maximum at the point B, with x-coordinate β.

Find the value of α, the value of β and the y-coordinates of the points A and B.

question continues on next page

(b) The functions g and h are defined as follows

$$g : x \to x + p \qquad x \in \mathbb{R}$$

$$h : x \to |x| \qquad x \in \mathbb{R}$$

where p is a constant.

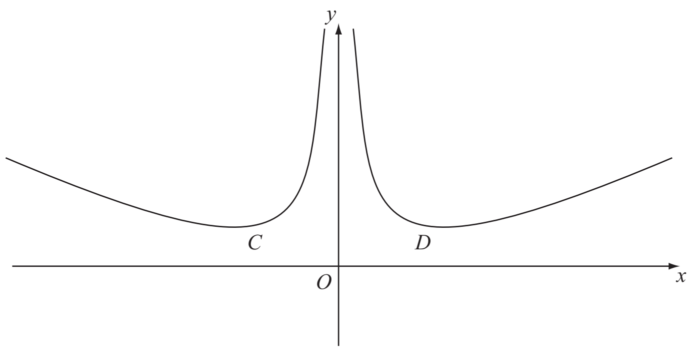


Figure 5

Figure 5 shows a sketch of the curve with equation $y = h(fg(x) + q)$, $x \in \mathbb{R}$, $x \neq 0$, where q is a constant. The curve is symmetric about the y-axis and has minimum points at C and D.

(i) Find the value of p and the value of q.

(ii) Write down the coordinates of D.

(5)

(c) The function m is given by

$$m(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R}, x \leqslant \alpha$$

where α is the x-coordinate of A as found in part (a).

(i) Find m^{-1}

(ii) Write down the domain of m^{-1}

(iii) Find the value of t such that $m(t) = m^{-1}(t)$

26. [2007 STEP I question 3]

Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence or otherwise evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta \,.$$

Evaluate also

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta \,.$$