1. Warm up

   (a) Show that
   $$\frac{x+11}{x^2+2x-3} \equiv \frac{3}{x-1} - \frac{2}{x+3}\,.$$

   (b) Evaluate A and B if
   $$\frac{5x+7}{x^2+3x+2} \equiv \frac{A}{x+1} + \frac{B}{x+2}\,.$$
   (Splitting a rational expression $f(x)$ as the sum of fractions with simpler denominators is known as expressing $f(x)$ in *partial fractions.*)

2. [2011 Oxford Admission Test question 1B] (Multiple Choice)

   A rectangle has perimeter P and area A. The values P and A must satisfy

   (a) $P^3 > A$ (b) $A^2 > P+1$ (c) $P^2 \geq 16A$ (d) $PA \geq A+P$

3. [2010 AEA question 2]

   The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$. Giving simplified answers in terms of p and q, find

   (a) the common difference of the terms in this series,

   (b) the first term of the series,

   (c) the sum of the first $(p+q)$ terms of the series.

4. [2007 STEP I question 2]

   (a) Given that $A = \arctan\frac{1}{2}$ and that $B = \arctan\frac{1}{3}$ (where A and B are acute) show, by considering $\tan(A+B)$, that $A+B = \frac{\pi}{4}$.

   The non-zero integers p and q satisfy
   $$\arctan\frac{1}{p} + \arctan\frac{1}{q} = \frac{\pi}{4}\,.$$
   Show that $(p-1)(q-1) = 2$ and hence determine p and q.

continued overleaf

(b) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan\frac{1}{r} + \arctan\frac{s}{s+t} = \frac{\pi}{4},$$

then there are only two possible values for t, and give r in terms of s in each case.

5. [2004 STEP I question 7]

(a) The function $f(x)$ is defined for $|x| < \frac{1}{5}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_0 = 2$, $a_1 = 7$ and $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$.

Simplify $f(x) - 7xf(x) + 10x^2 f(x)$, and hence show that $f(x) = \dfrac{1}{1-2x} + \dfrac{1}{1-5x}$.

Hence show that $a_n = 2^n + 5^n$.

(b) The function $g(x)$ is defined for $|x| < \frac{1}{3}$ by

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

where $b_0 = 5$, $b_1 = 10$, $b_2 = 40$, $b_3 = 100$ and $b_n = pb_{n-1} + qb_{n-2}$ for $n \geq 2$.

Obtain an expression for $g(x)$ as the sum of two algebraic fractions and determine b_n in terms of n.

6. [2010 AEA question 5]

$$I = \int \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x \quad x > 1$$

(a) Use the substitution $x = 1 + u^{-1}$ to show that

$$I = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c.$$

(b) Hence show that

$$\int_{\sec\alpha}^{\sec\beta} \frac{1}{(x-1)\sqrt{(x^2-1)}} = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right), 0 < \alpha < \beta < \frac{\pi}{2}.$$