The King's Factor

This file contains specific solutions to parts of some questions set in Year 13 tKF sessions, so that you can check that you are on the right track. Some questions, for instance those where the result is given and a proof is required, do not have solutions given here.

These brief solutions are NOT model answers and do not indicate how a full solution should be presented

Updated October 26, 2013

1. [2007 Oxford Admission Test question 1C] (Multiple Choice)

   If $y = \sin x$, then $2y^2 - 7y + 3 = 0$.

   This quadratic equation has solutions 3 and 1/2. There are no values of x for which $\sin x = 3$, and two values of x in the allowed range for which $\sin x = 1/2$.

   Answer (b)

2. [2007 Oxford Admission Test question 1F] (Multiple Choice)

   If $y = 2^x$ then $y^3 + 4 = y^2 + 4y$ and so $y^3 - y^2 - 4y + 4 = 0$.

   This means that $(y-1)y^2 - 4(y-1) = 0$ and so either $y - 1$ is zero or $y^2 - 4{=}0$.

   There are thus three solutions for y, $y = 1, 2$ or -2. However 2^x is positive if x is a real number, and so there are two solutions to the equation for x.

   Answer (c)

3. [2007 Oxford Admission Test question 1I] (Multiple Choice)

   If b is a positive number $(\log_{10} b)^2 > 0$.

   Hence $4(\log_{10} a)^2 < 1$ and so $\log_{10} a < 1/2$, $a < 10^{\frac{1}{2}} = \sqrt{10}$

   Answer (c)

4. [2007 Oxford Admission Test question 1J] (Multiple Choice)

   If $n \geq 1$ then

   $(n+1) + (n^4+2) + (n^9+3) + \cdots + (n^{10000}+100)$

   $\quad \geq (1+1) + (1^4+2) + (1^9+3) + \cdots + (1^{10000}+100)$

   Also $(1+1) + (1^4+2) + (1^9+3) + \cdots + (1^{10000}+100) = 2 + 3 + \cdots + 101 = 5150$.

   Hence $5150 > k$.

   Answer (d)

5. [2007 Oxford Admission Test question 3]

   (a) $I(c)$ is represented on a graph of $y = (x-1)^2 + 1$ by the area under the graph (and above the x axis) between $x = 0$ and $x = 1$.

   (b) $I(c) > 0$ for any value of c because $(c - x)^2 + c^2$ is never negative, and is zero either for for a single value of x (if $c = 0$) or for no values of x (if $c \neq 0$).

   (c) $I(c) = 2c^2 - c + \frac{1}{3}$.

   (d) Completing the square, $I(c) = 2(c - \frac{1}{2})^2 + \frac{5}{24}$. Hence the minimum value of $I(c)$ (as c varies) is $\frac{5}{24}$.

   (e) $-1 < \sin\theta < 1$. Since $I(c)$ has a minimum value when $c = \frac{5}{24}$, decreases as C grows from -1 to $\frac{5}{24}$ and increases as c grows from $\frac{5}{24}$ to 1, the maximum value of $I(\sin\theta)$ as θ varies is the maximum of $I(-1)$ and $I(1)$. (A good sketch of $I(c)$ will help you see this. Moral: draw a sketch even when you are not explicitly asked to do so.)

   The maximum value of $I(\sin\theta)$ as θ varies is $I(-1) = \frac{10}{3}$.

6. The gradient of the chord joining $(x+h, \frac{1}{x+h})$ and $(x, \frac{1}{x})$ where x and $x+h$ are both positive is

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{x + h - x} = \frac{-1}{x(x+h)}.$$

   Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{x+h-x} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

7. [2008 Oxford Admission Test question 1A] (Multiple Choice)

   $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 12x + 5$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $6x^2 - 12x + 5 = 0$.

   Since the discriminant for this quadratic equation is $(-12)^2 - 4 \times 6 \times 5$ which is positive there are two stationary points.

8. [2008 Oxford Admission Test question 1F] (Multiple Choice)

   Given that when the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,\mathrm{d}x$$

   by splitting the interval $0 \le x \le 1$ into 10 intervals then an overestimate of the integral is produced, it follows that the trapezium rule with 10 intervals also overestimates $\int_0^1 2f(x)\,\mathrm{d}x$, $\int_0^1 (f(x) - 1)\,\mathrm{d}x$ and $\int_1^2 f(x-1)\,\mathrm{d}x$ but underestimates $\int_0^1 (1 - f(x))\,\mathrm{d}x$ since the graph of this function is obtained from the graph of $f(x)$ by a reflection in a line parallel to the x-axis.

9. [2009 Specimen Paper 1 Oxford Admission Test question 1B] (Multiple Choice)

   $f'(x) = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$.

   Hence in the range $0 \le x \le 2$ the function $f(x) = 2x^3 - 9x^2 + 12x + 3$ has positive gradient when $0 < x < 1$, negative gradient when $1 < x < 2$ and zero gradient when $x = 1$ and $x = 2$. Also, the function has a local maximum when $x = 1$ and local minimum when $x = 2$.

   This shows that the smallest value of the function in the interval is either $f(0)$ or $f(2)$. (A sketch may help you to see that this must be the case.) Since $f(0) = 3$ and $f(2) = 7$, the smallest value of the function $f(x) = 2x^3 - 9x^2 + 12x + 3$ in the range $0 \le x \le 2$ is 3.

10. [2007 AEA question 2]

   (a) The graphs $y = x$ and $y = \sqrt{x}$, for $x \ge 0$, intersect at $(1,1)$ (and at $(0,0)$).

   (b) For $0 < x < 1$, $x < \sqrt{x}$ and so

   $$\int_0^1 x\,\mathrm{d}x < \int_0^a \sqrt{x}\,\mathrm{d}x$$

   while for $x > 1$, $\sqrt{x} < x$ and the difference $x - \sqrt{x}$ increases without limit as x increases so that for $a > 1$,

   $$\int_1^a x\,\mathrm{d}x > \int_1^a \sqrt{x}\,\mathrm{d}x\,.$$

   and the difference increases without limit as a increases. Hence there exists a value a of x ($a > 1$) such that

   $$\int_0^a x\,\mathrm{d}x = \int_0^a \sqrt{x}\,\mathrm{d}x\,.$$

   (c) $a = \frac{9}{4}$.

   (d) Translating the graphs $y = x$ and $y = \sqrt{x}$ by $-\frac{a}{2}$ in the x direction shows that,

   $$\int_{-b}^b f(x)\,\mathrm{d}x = \int_{-b}^b \sqrt{[f(x)]}\,\mathrm{d}x$$

   if $f(x) = x + \frac{a}{2}$ and $b = \frac{a}{2}$.
   Another possibility is to use $f(x) = |x|$ and $b = a$.

11. [2004 STEP I question 2]

(a) The graph of $y = \sqrt{[x]}$ is a series of horizontal line segments so that the integral can be calculated from first principles as

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}\,.$$

(b) Similarly

$$\int_0^a 2^{[x]}\,\mathrm{d}x = \sum_{r=0}^{a-1} 2^r = 2^a - 1\,.$$

(c) If a is positive but not an integer, $a = [a] + (a - [a])$ with $a - [a] > 0$. Hence

$$\int_0^a 2^{[x]}\,\mathrm{d}x = 2^{[a]} - 1 + (a - [a])2^{[a]}\,.$$

12. Warm up

(a) 92

(b) 19

13. [2008 Oxford Admission Test question 1G] (Multiple choice)

Since $4x - x^2 - 5 = -(x-2)^2 - 1$, $4x - x^2 - 5 \leq -1$. Also $4x - x^2 - 5 = -1$ when $x = 2$. As a result $y = \frac{1}{4x-x^2-5}$ is always negative and reaches a maximum value $\frac{1}{-1}$ when $x = 2$. The solution is thus (c).

14. [2008 Oxford Admission Test question 1D] (Multiple Choice)

When $1+3x+5x^2+7x^3+\cdots+99x^{49}$ is divided by $x-1$ the remainder is $1+3+5+\cdots+99 = 2500$. Hence (b) is correct.

15. [2008 Oxford Admission Test question 1J] (Multiple Choice)

Since $(3+\cos x)^2 \geq 4$ and $4 - 2\sin^8 x \leq 4$ the equation can only have solutions when both sides equal 4. In the range $0 \leq x < 2\pi$ this only occurs when $x = \pi$. Hence (b) is correct.

16. [2004 STEP I question 5]

General terms:

Writing A_0 for the first term in progression A, A_1 for the second etc we have $A_n = 5n + 1$. (Alternatively, if you write A_1 for the first term etc you have $A_n = 5n - 4$; this is valid, but slightly more complicated.)

With similar notation $B_n = 5n + 2$, $C_n = 5n + 3$, $D_n = 5n + 4$.

For the E series, let E_1 be the first term and you have $E_n = 5n$.

The square of every term in C is in D.

(i) -

(ii) -