THE KING'S FACTOR

This file contains specific solutions to parts of some questions set in Year 13 tKF sessions, so that you can check that you are on the right track. Some questions, for instance those where the result is given and a proof is required, do not have solutions given here.

These brief solutions are NOT model answers and do not indicate how a full solution should be presented

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1. Warm up

   (a) -

   (b) $A = 2$ and $B = 3$.

2. [2011 Oxford Admission Test question 1B] (Multiple Choice)

   With the notation used in the hint, $P = 2(x+y)$, hence $A = xy = x(\frac{P}{2} - y)$ and so $A \leq \frac{P^2}{16}$.

   The solution is (c) $P^2 \geq 16A$

3. [2010 AEA question 2]

$$
\begin{aligned}
q &= \frac{p}{2}\,(2a + p - 1)d) \\
p &= \frac{q}{2}\,(2a + q - 1)d)
\end{aligned}
$$

   (a) Eliminating a between these two equations gives $d = -\dfrac{2(p+q)}{pq}$.

   (b) $a = \dfrac{q^2 + pq + p^2 - p - q}{pq}$. (There are other equally simple correct expressions.)

   (c) Let t_r denote the r^{th} term. To evaluate the sum of the first $(p+q)$ terms, first compare $t_{p+1} + \ldots t_{p+q}$ to $t_1 + \cdots + t_q$.

   Since $t_{p+1} = t_1 + pd$, $t_{P+2} = t_2 + pd$ etc

   $t_{p+1} + \ldots t_{p+q} = (t_1 + \cdots + t_q) + qpd = p + qpd.$

   Hence the sum of the first $p + q$ terms is $p + q + qpd = -(p+q)$.

4. [2007 STEP I question 2]

   (a) The only possible non-zero integer values of p and q which satisfy $(p-1)(q-1)=2$ are $p=2$ and $q=3$ (or vice versa).

   (b) $r=\frac{2s}{t}+1$.
   Since r is an integer, either $t=2$ (in which case $r=s+1$) or $t=1$ (in which case $r=2s+1$).

5. [2004 STEP I question 7] [2004 STEP I question 7]

   (a)
   $$f(x)-7xf(x)+10x^2f(x)=2-7x$$
   and hence
   $$f(x)=\frac{2-7x}{10x^2-7x+1}=\frac{1}{1-2x}+\frac{1}{1-5x}\,.$$
   (Hence $a_n=2^n+5^n$.)

   (b) $p=1$ and $q=6$.
   $$g(x)-xg(x)-6x^2g(x)=5+5x$$
   and so
   $$g(x)=\frac{5+5x}{1-x-6x^2}=\frac{1}{1+2x}+\frac{1}{1-3x}\,.$$
   $b_n=(-2)^n+4(3)^n$.

6. [2010 AEA question 5]
   $$I=\int\frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x \quad x>1$$

   (a) Using the substitution $x=1+u^{-1}$
   $$I=\int\frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}=\int\frac{-1}{\sqrt{2u+1}}\,\mathrm{d}u=-\sqrt{2u+1}+c=-\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}+c\,.$$

   (b) Let $t=\tan\dfrac{\alpha}{2}$. Then $\tan\alpha=\dfrac{2t}{1-t^2}$ and so $\sec\alpha=\dfrac{1+t^2}{1-t^2}$.
   Hence $\dfrac{\sec\alpha+1}{\sec\alpha-1}=\dfrac{1}{t^2}$, hence result.