The King's Factor

This file contains hints for questions set in Year 13 tKF sessions, to give you a nudge if you feel really stuck

Updated October 26, 2013

1. [2007 Oxford Admission Test question 1C] (Multiple Choice)

   Let $y = \sin x$ and construct a quadratic equation which y must satisfy if x satisfies the equation
   $$7 \sin x + 2 \cos^2 x = 5 \, .$$

2. [2007 Oxford Admission Test question 1F] (Multiple Choice)

   Let $y = 2^x$ and construct a cubic equation which y must satisfy if x satisfies the equation
   $$8^x + 4 = 4^x + 2^{x+2} \, .$$
   You will find you can find one solution to this cubic by inspection, that is, by looking at the equation, without needing to do any long calculation.

3. [2007 Oxford Admission Test question 1I] (Multiple Choice)

   To answer this question about a you need to consider all possible values of $(\log_{10} b)^2$. This will give you an upper limit for $4(\log_{10} a)^2$ and hence for a.

4. [2007 Oxford Admission Test question 1J] (Multiple Choice)

   First decide how $(n + 1) + (n^2 + 2) + (n^3 + 3) + \cdots + (n^{10000} + 100)$ will vary as n varies.

   Given that $n \geq 1$, does this expression have a least value or a greatest value? If so what is it?

5. [2007 Oxford Admission Test question 3]

Let

$$I(c) = \int_0^1 ((c-x)^2 + c^2)\,\mathrm{d}x$$

where c is a real number.

(a) First use standard methods to sketch the graph of $y = (x-1)^2 + 1$.

(b) What simple condition on the integrand is a sufficient condition for an integral to be positive?

(c) Use direct calculation, being careful to remember that c is a constant and x the variable of integration.

(d) You will of course need to have the correct solution to (c) to answer this. Although you can use calculus, $I(c)$ is a quadratic in c, which means that completing the square will allow you to sketch a graph of $I(c)$ against c, which helps to find a a global minimum and not just a local one.

(e) As $I(c)$ does not have a local maximum you might be surprised by this question. But remember that $\sin x$ can only take a restricted range of values, and use the graph recommended in part (d).

6. If differentiation from first principles was new to you, or almost forgotten, then look at the introduction to differentiation given in any good A-level textbook.

To prove from first principles that if $y = \dfrac{1}{x}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x^2}$, start by sketching the curve $y = \dfrac{1}{x}$. Mark on your sketch the points $(x, \frac{1}{x})$ and $(x+h, \frac{1}{x+h})$ for suitable positive values of x and h such as 2 and $\frac{1}{2}$. Then follow the instructions in the question.

7. [2008 Oxford Admission Test question 1A] (Multiple Choice)

There are no tricks in this question! A function has a stationary point when its first derivative is zero. (A stationary point can be a local minimum, a local maximum or a point of inflexion.)

8. [2008 Oxford Admission Test question 1F] (Multiple Choice)

   To answer this question you MUST sketch some graphs.

   First, sketch a function such that the trapezium rule will overestimate its integral over the interval $0 \le x \le 1$. (You can use 4 or 5 intervals, say, to make your sketch a bit simpler.) You may need to try a few different curves before you see the characteristics of functions whose integrals are overestimated by the trapezium rule.

   Next, having sketched a function $f(x)$ whose integral over the interval $0 \le x \le 1$ is overestimated by the trapezium rule, sketch $2f(x)$, $f(x) - 1$, $f(x-1)$ and $1 - f(x)$.

9. [2009 Specimen Paper 1 Oxford Admission Test question 1B] (Multiple Choice)

   Here do not use calculus methods in an unthinking way; you must sketch the graph of

   $f(x) = 2x^3 - 9x^2 + 12x + 3$ in the range $0 \le x \le 2$.

10. [2007 AEA question 2]

    (a) Start by sketching $y = x$. To sketch $y = \sqrt{x}$ consider a few specific values of x such as $x = \frac{1}{2}$, $x = 1$, $x = 4$.

    (b) Explain why

    $$\int_0^1 x \,\mathrm{d}x < \int_0^1 \sqrt{x} \,\mathrm{d}x \,.$$

    is negative. Also explain why

    $$\int_1^a x \,\mathrm{d}x > \int_1^a \sqrt{x} \,\mathrm{d}x$$

    is positive when $a > 1$, and increases without limit as a increases.

    (c) Use direct calculation to find the exact value of a.

    (d) There are various possibilities, one of which is to shift the graph in the negative x direction.

11. [2004 STEP I question 2]

 (a) The graph of $y = \sqrt{[x]}$ is a series of horizontal lines. This means that

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x$$

 (which is the area under the graph) can be calculated directly as a finite sum.

 (b) This too can be calculated as a finite sum.

 (c) Here consider $a = [a] + (a - [a])$.

12. Warm up

 (a) To evaluate $1+4+7+\cdots+22$ use the formula for summing an arithmetic progression. Or use the method which derives this formula which is to note that evaluating

$$\begin{array}{rl} & 1 + 4 + 7 + \ \ldots\ +16 + 19 + 22 \\ + & 22 + 19 + 16 + \ \ldots\ +7 + 4 + 1 \end{array}$$

 is much easier and gives twice the sum required.

 (b) By the remainder theorem, the remainder when $P(x) = x^3 + 3x^2 - 4x + 7$ is divided by $x + 2$ is $P(-2)$.

13. [2008 Oxford Admission Test question 1G] (Multiple choice)

 Sketch the graph of $y = 4x - x^2 - 5$.

14. [2008 Oxford Admission Test question 1D] (Multiple Choice)

 Use the remainder theorem to give an expression for the remainder when

$$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

 is divided by $x - 1$.

15. [2008 Oxford Admission Test question 1J] (Multiple Choice)

 Use the fact that $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ to find the range of possible values of $(3 + \cos x)^2$ and the range of possible values of $4 - 2\sin^8 x$.

16. [2004 STEP I question 5]

See solutions for general terms.

You need algebraic working such as:

$$B_n + C_m = 5n + 2 \quad + \quad 5m + 3 = 5(n + m) + 5 = 5(m + n + 1)$$

which is a term in E.

See solutions for progression in which every term in C must lie.

(i) First explain why 243 723 is in C.
Next explain why x^2 and $x^2 + 5y$ must lie in the same progression.
Finally, investigate in turn where the square of a term of A must lie, where the square of a term of B must lie etc.

(ii) Here you need the same kind of argument, but it is rather more complicated.