This file contains hints for questions set in Year 12 tKF sessions, to give you a nudge if you feel really stuck

Updated February 20, 2014

1. To calculate the square root of 24×150 break 24 and 150 down into factors until all factors are either prefect squares or occur twice.

   Further hint: $24 \times 150 = 4 \times 6 \times 6 \times 25$.

   Alternatively, you can factorise 24×150 into prime factors as $2 \times 2 \times 2 \times 3 \quad \times \quad 2 \times 3 \times 5 \times 5 = 2^4 \times 3^2 \times 5^2$.

2. This can be proved by induction.

   Alternatively, note that any integer n can be expressed as either $3m$ or $3m+1$ or $3m+2$ for some integer m.

   Taking, for example, $n = 3m+1$, show that $(3m+1)^3 - (3m+1)$ is a multiple of 3. (Repeat for the other two cases.)

3. [2007 Oxford Admission Test question 1B] (Multiple Choice)

   First decide the range of possible values of $\sin^2(10x+11)$.

   Use this to decide the range of possible values of $3\sin^2(10x+11)^2 - 7$.

   And so on.

4. [2008 Oxford Admission Test question 1E] (Multiple Choice)

   Try this a bit at a time, until you can see how the highest power will emerge:

   What is the highest power of x in

   $$f(x) = \left(2x^6+7\right)^3\,?$$

   What is the highest power of x in

   $$f(x) = \left(2x^6+7\right)^3 + \left(3x^8-12\right)^4\,?$$

   What is the highest power of x in

   $$f(x) = \left[\left(2x^6+7\right)^3 + \left(3x^8-12\right)^4\right]^5\,?$$

etc etc

5. [2008 Oxford Admission Test question 2]

   (a) To find a pair of positive integers x_1 and y_1, that solve the equation
   $$(x_1)^2 - 2(y_1)^2 = 1$$
   simply try out some small integers.

   (b) Substitute
   $$x_{n+1} = 3x_n + 4y_n \quad \text{and} \quad y_{n+1} = ax_n + by_n$$
   into
   $$(x_n)^2 - 2(y_n)^2 = (x_{n+1})^2 - 2(y_{n+1})^2 \,.$$
   and match coefficients.

   (c) Using your values of x_1, y_1 from part (a), and the recurrence relations
   $$x_{n+1} = 3x_n + 4y_n \quad \text{and} \quad y_{n+1} = ax_n + by_n$$
   (with values of a and b from part (b)) evaluate x_2 and y_2 and explain why they satisfy
   $$(x_2)^2 - 2(y_2)^2 = 1 \,.$$
   Repeat this to calculate x_3, y_3 and so on until both exceed 50.

   (d) Rearrange the equation $(x_n)^2 - 2(y_n)^2 = 1$. (Why does this equation hold?)

6. [2004 STEP I question 3]

   (a) To show that $x - 3$ is a factor of
   $$g(x) = x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y \,. \qquad (*)$$
   show that $g(3) = 0$ and apply the factor theorem.
   Next note that
   $$g(x) = x^3 - 5x^2 + 6x + y(2x^2 - 8x + 6) + y^2(x - 3)$$
   and factorise $x^3 - 5x^2 + 6x$ and $2x^2 - 8x + 6$.

   (b) To factorise $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ first find a factor of the form $y + c$.

7. For each of the following statements *either* prove that it is true *or* give a counter-example to show that it is false:

   (a) The product of any two even numbers is a multiple of 4.
   Hint: use the fact that any two even numbers can be written $2m$ and $2n$ where m and n are integers.

   (b) The product of any two even numbers is a multiple of 8.
   Hint: Can you find a counter-example?

   (c) The product of any two odd numbers is a multiple of 3.
   Hint: Can you find a counter-example?

   (d) The product of any two odd numbers is and odd number.
   Hint: use the fact that any two odd numbers can be written $2m+1$ and $2n+1$ where m and n are integers.

8. Question

   (a) Show that if
   $$x^2 + y^2 - 4x + 10y + 4 = 0 \qquad (*)$$
   then the distance of the point (x, y) from the point $(2, -5)$ is equal to 5.

   (b) Sketch the set of points which satisfy $(*)$ (i.e. sketch the curve $x^2+y^2-4x+10y+4 = 0$.)

   (c) Find the equation of the tangent to this curve at the point $(5, -1)$.

   Hints

   (a) Show that the distance between the points (x, y) and $(2, -5)$ is equal to
   $$x^2 + y^2 - 4x + 10y + 4 + 25 = 0\,.$$

   (b) By arguments used in solving (a), the set of points which satisfy $(*)$ is the set of points whose distance from $(2, -5)$ is 5.

   (c) You do not need calculus here, although it is can be used.
   Use the fact that the radius through a point P on a circle is perpendicular to the tangent to the circle at P.

9. [2009 Specimen paper Oxford Admission Test question 1A]

   Draw a rough graph including coordinate axes and the points P and Q. Then consider how far in the x direction would take you a third of the way from P to Q, and similarly for the y direction.

10. [2009 Specimen paper Oxford Admission Test question 1D]

    Sketch, on the same axes, the regions where

$$\begin{aligned} 2x + 3y &> 23 \\ x + 2 &> 3y, \\ 3y + 1 &> 4x. \end{aligned} \tag{1}$$

    This should suggest the anwswer.

    Then use algebra to find an upper limit for x.

    Finally, show that it is possible for x to take this value by finding a value for y as well and checking that all three inequalities are satisfied.

11. [2009 AEA question 1]

    (a) To sketch the second curve, remember that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x \leq 0$.

    (b) Solve two different equations, one when $x \geq 0$ and the other when $x \leq 0$.

12. [2006 STEP I question 1]

    To find the integer n that satisfies $n^2 < 33\,127 < (n+1)^2$, make an estimate of $\sqrt{33\,127}$.

    For the final part, remember the factorisation of the difference of two squares.

13. (a) First note that $\dfrac{4^5 \times 5^{-3}}{2^8 \times 10^{-4}} = \dfrac{4^5 \times 10^4}{2^8 \times 5^3}$.
    Then express numerator and denominator as powers of 2 and 5, and cancel down where possible.

    (b) To express $\dfrac{(1+\sqrt{7})^2}{3+\sqrt{7}}$ in the form $a + b\sqrt{7}$, first multiply by $1 = \dfrac{3-\sqrt{7}}{3-\sqrt{7}}$

14. [2010 Oxford Admission Test question 1A] (Multiple choice) Write the quadratic equation which the x-coordinate of any point of intersection must satisfy in standard form. Use the discriminant (i.e. $b^2 - 4ac$ in standard notation) to find a condition which k must satisfy if the line intersects the curve.

    (Also sketch $y = (x-1)^2$ to see the sort of condition k must satisfy for the line to intersect the curve.)

15. [2004 AEA question 3]

$$f(x) = x^3 - (k+4)x + 2k \quad \text{where k is a constant.}$$

    (a) Substitute a particular value of x into the formula for $f(x)$.

    (b) Use (a) to show that one factor of $f(x)$ is $(x-2)$.
    Hence find a and b such that $f(x) = (x-2)(x^2 + ax + b)$.
    Then think of two ways in which you can get two equal roots for $f(x) = 0$.

    (c) Sketch $y = f(x)$ for the positive value of k you found in (b). This should show you the link between equal roots and the x-axis being a tangent to the curve.

16. [2007 Oxford Admission Test question 1A] (Multiple Choice)

    Begin by expressing

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

    in the form $2^a \times 3^b$.

    This is possible because each of $6, 12, 8$ and 9 contains only factors of 2 and 3.

17. [2000 STEP I question 6]

    To sketch the region of the x-y plane where $x^2-y^2+x+3y>2$ first sketch the lines $x-y+2=0$ and $x+y-1=0$.

    To sketch the region of the x-y plane for which $x^2-4y^2+3x-2y<-2$ first factorise $x^2-4y^2+3x-2y+2$ into two linear factors.

18. [2004 STEP I question 1]

    (i) You can work this out in two stages, first calculating $(3+2\sqrt{5})^2$ and then multiplying by $(3+2\sqrt{5})$. You can also work it out using the binomial theorem, but it is probably worth learning the first few rows of Pascal's triangle, including the row

    $$1 \quad 3 \quad 3 \quad 1\,,$$

    which immediately gives you $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

    (ii) Use the condition $\sqrt[3]{99-70\sqrt{2}} = c-d\sqrt{2}$ to write down two simultaneous equations satisfied by c and d. Rather than trying to solve these equations analytically, read the question carefully [IMPORTANT PRINCIPLE: make sure you use all the information given in a question] and notice that c and d are positive integers. Look at the equations until you see that there are very limited possibilities for c and d, and then use trial and error. (Make sure both equations are satisfied.)

    (iii) This looks daunting as it has x^6 in it, but the only other power is x^3 so you can set $y=x^3$ giving the quadratic equation $y^2-198y^3+1=0$. Solve this, to get $y=99\pm70\sqrt{2}$. Then use the previous part to solve for x.

19. Warm up:

    To do this without calculus, complete the square.

    With calculus, find the value of x for which the gradient is zero.

20. [2009 Specimen 2 Oxford Admission Test question 1F] (Multiple Choice)

    Use completion of the square (or calculus) to find the coordinates of the turning point of the parabola

    $$y = x^2 - 2ax + 1$$

    in terms of a.

    Then write down an expression for the square of the distance of this point from $(0, 0)$.

    Find the values of a which make this a minimum.

21. [2008 Oxford Admission Test question 1G] (Multiple choice)

    Sketch the graph of $y = 4x - x^2 - 5$.

22. [2007 Oxford Admission Test question 1E] (Multiple choice)

    Substitute particular values of x and n into

    $$(1 - x)^n(2 - x)^{2n}(3 - x)^{3n}(4 - x)^{4n}(5 - x)^{5n}$$

    and consider whether the resulting value of the expression will be positive, negative or zero.

23. [2009 Specimen 2 Oxford Admission Test question 1G] (Multiple Choice)

    This question requires no more advanced mathematics than the 13 times table!

    Read the question carefully, and then start to write down a number starting with 9 which has the required property.

24. Warm up

(a) To work out the values of $g(x-3)+7$ think carefully what this expression means. For a start, if $g(x)=x^2$, $g(u)=u^2$, $g(x+11)=(x+11)^2$, $g(\frac{\alpha}{17})=\left(\frac{\alpha}{17}\right)^2$ and so on.

(b) As in (a) you have to think carefully what the expression $g(x-3)+7$ means.

(c) You probably already know the turning point of $y=g(x)=x^2$.
To find the turning point of $y=g(x-3)+7$ there are certainly three approaches. Ideally you should try them all.
Three approaches:
- differentiation,
- using the fact that $a^2 \geq 0$ and $a^2=0$ if and only if $a=0$
- using relationship between the graphs of $y=g(x)$ and $y=g(x-3)+7$

For the sketches, you will probably know what the graph of $y=g(x)$ looks like, and can then sketch the graph of $y=g(x-3)+7$ by applying the appropriate transformation.

25. (a) Here you need to think of the translations you would apply to the graph of $y=f(x)$ to obtain each of the graphs $y=f(x-3)$, $y=f(x-3)+7$ and $y=f(x+4)-8$.

(b) Here you have to think a bit harder. Begin by sketching the portion of $y=f(x)$ for values of x around 3.

i. Now add to your graph a sketch of $y=-f(x)$ for values of x around 3.

ii. For this part, you need to add a sketch of $y=\frac{1}{f(x)}$ for values of x around 3. First work out the value of y when $x=3$.

26. [2009 Oxford Admission Test question 1C]

Here you might well first try rearranging to get $x^4-x^2+2cx-c^2=0$. While this is not incorrect it does not make the equation easier to solve.

It is better to notice that both sides of the equation are squares, and use the fact that

$$a^2=b^2 \quad \text{if and only if} \qquad \textit{either} \quad a=b \quad \textit{or} \quad a=-b\,.$$

27. [2007 Oxford Admission Test question 1G]

    Here it is best to proceed by elimination. Two of the possibilities are fairly easily eliminated, one more is eliminated if you think carefully about the values of x for which the function will be zero.

28. [2005 STEP I question 6]

    (a) You need to use the formula for the distance between two pints whose coordinates are given.

    If you can't remember it, look it up, learn it (it is one you need to know in your sleep) and make sure you know how it is proved using Pythagoras' theorem.

    Use the formula to write down the square of the distance AP and the square of the distance BP.

    Then use the condition $AP = 2BP$ to build an equation which the x and y coordinates of P must satisfy.

    (b) Here proceed similarly to build the equation of the path of Q.

    Compare the two equations. If the the path of Q is the same as the path of P, the two equations must be the same except that one may be a multiple of the other by a non-zero factor.

    This gives you equations relating a, b and k.

    By eliminating k you should eventually show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}.$$