Updated February 20, 2014

This file contains specific solutions to parts of some questions set in Year 12 tKF sessions, so that you can check that you are on the right track. Some questions, for instance those where the result is given and a proof is required, do not have solutions given here.

These brief solutions are NOT model answers and do not indicate how a full solution should be presented

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1. $\sqrt{24 \times 150} = 2 \times 3 \times 5 = 60$
   $\sqrt{147 \times 48} = 84$
   $\sqrt{abc^2 \times c^4 ab} = abc^3$

2. -

3. [2007 Oxford Admission Test question 1B] (Multiple Choice)
   (d) 49, achieved when $\sin(10x + 11) = 0$.

4. [2008 Oxford Admission Test question 1E] (Multiple Choice)
   (d) 504

5. [2008 Oxford Admission Test question 2]
   (a) The smallest positive integers x_1 and y_1, that solve the equation
   $$(x_1)^2 - 2(y_1)^2 = 1$$
   are $x_1 = 3$ and $y_1 = 2$. (There is an infinite number of integral solutions to these equations.)
   (b) $a = 2$, $b = 3$
   (c) The integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$ are $X = 99$, $Y = 70$. (There is an infinite number of larger possibilities.)
   (d) (Using the values of a and b found in part (b)) $\dfrac{x_n}{y_n} \sim \sqrt{2}$ as n increases.
   (In more formal mathematical terms, $\dfrac{x_n}{y_n} \to \sqrt{2}$ as $n \to \infty$. This statement can be proved using a branch of mathematics known as analysis.)

6. [2004 STEP I question 3]

   (a)
   $$x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y = (x-3)(x+y)(x+y-2)$$

   (b)
   $$6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10 = (y+2)(2y-x-1)(3y-x-5)$$

7. For each of the following statements *either* prove that it is true *or* give a counter-example to show that it is false:

   (a) The product of any two even numbers is a multiple of 4.
   True. Express the two even numbers as $2m$ and $2n$ where m and n are integers.
   Then note that $2m \times 2n = 4(mn)$.

   (b) The product of any two even numbers is a multiple of 8.
   False. Counter-example: $2 \times 6 = 12$.

   (c) The product of any two odd numbers is a multiple of 3.
   False. Counter-example: $5 \times 7 = 35$.

   (d) The product of any two odd numbers is and odd number.
   True. Express the two odd numbers as $2m+1$ and $2n+1$.
   Then note that $(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$.

8. (a) The square of the distance of the point (x, y) from the point $(2, -5)$ is equal to

$$(x-2)^2 + (y+5)^2 = x^2 + y^2 - 4x + 10y + 4 + 25\,.$$

   Hence if

$$x^2 + y^2 - 4x + 10y + 4 = 0 \qquad (*)$$

   then the square of this distance is 25 and so the distance is 5.

   (b) The set of points which satisfy $(*)$ is the circle with centre $(2, -5)$ and radius 5.

   (c) The gradient of the radius of the circle through $(5, -1)$ is $\dfrac{4}{3}$. Hence the equation of the tangent to this curve at the point $(5, -1)$ is

$$4x + 3y = 11\,.$$

9. [2009 Specimen paper Oxford Admission Test question 1A]
   (d) $(4, 1)$

10. [2009 Specimen paper Oxford Admission Test question 1D]
   (d) $x \leq 7$, with $x = 7, y = 3$ satisfying all three conditions.

11. [2009 AEA question 1]

    (a) The curve $y = (x+1)(2-x)$ crosses the x-axis at $(-1, 0)$ and $(2, 0)$ and the y-axis at $(0, 2)$.
    The curve $y = x^2 - 2|x|$ crosses the x axis at $(2, 0)$ and $(-2, 0)$ and the x and y axes at $(0, 0)$.

    (b) The two curves intersect where $x = 2$ and $x = \dfrac{-1-\sqrt{17}}{4}$.

12. [2006 STEP I question 1]

    The integer, n, that satisfies $n^2 < 33\,127 < (n+1)^2$ is $n = 182$.

    Since $184^2 - 33\,127 = 27^2$ a small integer m such that $(n+m)^2 - 33\,127$ is a perfect square is $m = 2$.

    Since $184^2 - 33\,127 = 27^2$, Since $184^2 - 27^2 = 33127$ and so $33\,127 = 211 \times 157$.

13. Warm up:

    (a) 320

    (b) $5 - \sqrt{7}$.

14. [2010 Oxford Admission Test question 1A] (Multiple choice)

    (c) $k \geq 0 \quad \text{or} \quad k \leq -4$

15. [2004 AEA question 3]

$$f(x) = x^3 - (k+4)x + 2k \quad \text{where k is a constant.}$$

   (a) $f(2) = 0$ for all k.

   (b) From the first part we know one root is $x = 2$ and hence that $x - 2$ is a factor of $f(x)$. Either by inspection or by long division,

$$f(x) = x^3 - (k+4)x + 2k = (x-2)(x^2 + 2x - k)\,.$$

   Hence either $x = 2$ or $x^2 + 2x - k = 0$.
   Equal roots for $f(x) = 0$ will occur if the quadratic equation has equal roots, i.e. if $k = -1$.
   Equal roots for $f(x) = 0$ will also occur of one of the roots of the quadratic is 2, i.e. if $2^2 + 2 \times 2 - k = 0$, i.e. if $k = 8$.

   (c) Since the x-axis, ie the line $y = 0$, is a tangent to the curve with equation $y = f(x)$, the equation $f(x) = 0$ must have two equal roots.
   Since k is positive, we know from (b) that $k = 8$.
   A sketch of $y = f(x)$ when $k = 8$ shows that the line $y = p$ intersects the curve in three distinct points if p is positive and less than the local maximum value of y.
   This local maximum occurs when $f'(x) = 0, x \neq 2$, ie when $x = -2$ and $f(x) = 32$.
   Hence $0 < p < 32$.

16. [2007 Oxford Admission Test question 1A] (Multiple Choice)

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}} = 2^{-s} \times 3^{-4s}$$

   which is an integer if $s \leq 0$, hence (b) is the correct answer.

17. [2000 STEP I question 6]

The region of the x-y plane where

$$x^2 - y^2 + x + 3y > 2$$

is the region where *either* both $x - y + 2 > 0$ and $x + y - 1 > 0$ *or* both $x - y + 2 < 0$ and $x + y - 1 < 0$.

The sketch should include the lines $x - y + 2 = 0$ and $x + y - 1 = 0$ intersecting at $(-\frac{1}{2}, \frac{3}{2})$. These lines divide the plane into four regions which can be labelled left, right, upper and lower. Of these four regions, the left and right regions together make up the region of the x-y plane where

$$x^2 - y^2 + x + 3y > 2.$$

Since $x^2 - 4y^2 + 3x - 2y + 2 = (x - 2y + 1)(x + 2y + 2)$ the region of the x-y plane for which

$$x^2 - 4y^2 + 3x - 2y < -2$$

is the region where *either* $x - 2y + 1 > 0$ while $x + 2y + 2 < 0$ *or* $x - 2y + 1 < 0$ while $x + 2y + 2 > 0$.

The sketch should include the lines $x - 2y + 1 = 0$ and $x + 2y + 2 = 0$ intersecting at $(-\frac{3}{2}, \frac{1}{4})$. These lines divide the plane into four regions which can be labelled left, right, upper and lower. Of these four regions, the upper and lower regions together make up the region of the x-y plane where

$$x^2 - y^2 + x + 3y > 2.$$

A point for which both inequalities are satisfied is $(1, 2)$. (There are many other possible points.)

18. [2004 STEP I question 1]

    (i) $207 + 94\sqrt{5}$.

    (ii) $c = 3$ and $d = 2$.

    (iii) $x = 3 \pm 2\sqrt{2}$.

19. Warm up: Using algebra (and no calculus):

    Completing the square shows $f(x) = x^2 - 6x + 13 = (x - 3)^2 + 4$ and hence that $f(x) \geq 4$.

    Using calculus:

    If $y = x^2 - 6x + 13$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 3$, $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ when $x < 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ when $x > 3$.

    Also, when $x = 3$, $y = 4$. Hence the minimum value of $f(x)$ is 4.

    The graph of $y = f(x)$ is a parabola with minimum point $(3, 4)$.

    (If you can, do this in two ways, one using calculus and the other not using calculus.)

20. [2009 Specimen 2 Oxford Admission Test question 1F] (Multiple Choice)

The turning point of the parabola

$$y = x^2 - 2ax + 1$$

is at $(a, 1 - a^2)$.

If d is the distance of this turning point from O, then

$$d^2 = a^2 + (1 - a^2) = b - (1 - b)^2 \quad \text{if} \quad b = a^2\,.$$

Since $b - (1-b)^2 = (b - \frac{1}{2})^2 + \frac{3}{4}$, d^2 has its a minimum value $\frac{3}{4}$ when $b = a^2 = \frac{1}{2}$. Hence $a = \pm\frac{1}{\sqrt{2}}$ and the answer is (d)

21. [2008 Oxford Admission Test question 1G] (Multiple choice)

Since $4x - x^2 - 5 = -(x-2)^2 - 1$, $4x - x^2 - 5 \leq -1$. Also $4x - x^2 - 5 = -1$ when $x = 2$.

As a result $y = \frac{1}{4x - x^2 - 5}$ is always negative and reaches a minimum value $\frac{1}{-1}$ when $x = 2$.

The solution is thus (c).

22. [2007 Oxford Admission Test question 1E] (Multiple choice)

    If $x = 4$ then $(1-x)^n(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n} = 0$ which eliminates (a) and (d).

    If $n = 6$, $(1-x)^n(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n} \geq 0$ which eliminates (c).

    Finally, to show (b) is true, note that if $x > 5$ every factor is negative and also if n is odd there is an odd number of factors.

23. [2009 Specimen 2 Oxford Admission Test question 1G] (Multiple Choice)

    Using the 13 tines table up to $7 \times 13 = 91$ shows that the number must be of the form $913913913\ldots$.

    Since $100 = 33 \times 3 + 1$ the nuber must end in 9 and so (d) is correct.

24. Warm up

    (a)

x	-6	-4	-2	0	2	4	6
$g(x)$	36	16	4	0	4	16	36
$g(x-3)+7$	88	56	32	18	8	8	16

    (b) $g(x-3)+7 = (x-3)^2+7 = x^2-6x+16$

    (c) The turning point of $y = g(x)$ is a minimum at $(0,0)$.
    The turning point of $y = g(x-3)+7$ is a minimum at $(3,7)$.

25. (a) i. The graph of $f(x-3)$ has a minimum point at $(6,2)$.
    ii. The graph of $f(x-3)+7$ has a minimum point at $(6,9)$.
    iii. The graph of $f(x+4)-8$ has a minimum point at $(-1,-1)$.

    (b) i. The graph of $-f(x)$ has a maximum point at $(3,-2)$.
    ii. The graph of $\frac{1}{f(x)}$ has a maximum point at $(3, \frac{1}{2})$.

26. [2009 Oxford Admission Test question 1C]

    Taking square roots, $x^4 = (x-c)^2$ if either $x^2 = +(x-c)$ or $x^2 = -(x-c)$.

    $x^2 = x - c$ is a quadratic equation which has solutions if $-\frac{1}{4} \leq c$.

    $x^2 = -(x-c)$ is a quadratic equation which has solutions if $c \leq \frac{1}{4}$.

    Hence $x^4 = (x-c)^2$ has four real solutions (including possible repeated roots) for $-\frac{1}{4} \leq c \leq \frac{1}{4}$.

    The correct answer is (b).

27. [2007 Oxford Admission Test question 1G]

    Not (b), as function never takes negative values, not (d) as function has value 0 when $x = 0$. Not (c) because zeros will become more frequent as x increases. Hence (a) must be correct.

28. [2005 STEP I question 6]

    (a) -

    (b) If
    $$QC = k \times QD\,,$$
    then
    $$(k^2-1)x^2 + (k^2-1)y^2 + 2(a-k^2b)x + k^2b^2 - a^2 = 0\,.$$
    The curve in part (a) has equation
    $$x^2 + y^2 + 14x - 51 = 0\,.$$
    These are equations defining the same curve if
    $$\frac{k^2-1}{1} = \frac{2(a-k^2b)}{14} = \frac{k^2b^2-a^2}{-51}\,.$$
    Eliminate k between these two equations to get result.
    Last part: rearrange by putting all terms on one side to get
    $$ab^2 - a^2b + 7(b^2-a^2) + 51(a-b) = 0\,.$$
    Dividing through by $b-a$ which is possible when $a \neq b$ then gives
    $ab + 7(b+a) - 51 = 0$ and hence $(a+7)(b+7) = 100$.