1. [2009 STEP I question 2]

   First calculate $\frac{dy}{dx}$ in terms of y and x using implicit differentiation. Use the result to calculate the equation to the tangent to the curve at $(-a, b)$. (Check that $(-a, b)$ does lie on the curve.)

   To show that in the case $a = 1$ and $b = 2$ the x-coordinates of the points where the tangent meets the curve satisfy

   $$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

   write down the equation of the curve and the tangent when $a = 1$ and $b = 2$, and use substitution to eliminate y.

   Solve the equation

   $$7x^3 - 3x^2 - 27x - 17 = 0$$

   using the fact that one of the roots is already known. Use the positive root of the equation to find positive integers p, q, r and s such that

   $$p^3 = q^3 + r^3 + s^3\,.$$

2. [2007 STEP I question 3]

   To answer this question you need to have in mind the various identities relating $\cos\theta$, $\sin\theta$, $\cos 2\theta$ and $\sin 2\theta$.

   For example, to obtain the expression for $\cos^4\theta + \sin^4\theta$, play around with

   $(\cos^2\theta + \sin^2\theta)^2 = 1$.

   Once you have proved the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$ you should be able to evaluate

   $$\int_0^{\frac{\pi}{2}} \cos^4\theta - \sin^4\theta \,d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^4\theta + \sin^4\theta \,d\theta$$

   and hence (with very little more work) obtain the two required integrals.

   To evaluate

   $$\int_0^{\frac{\pi}{2}} \cos^6\theta \,d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^6\theta \,d\theta\,.$$

   take a hint from the first part: find expressions for $\cos^6\theta - \sin^6\theta$ and $\cos^6\theta + \sin^6\theta$ which you can integrate.

3. [1998 STEP I question 5]

   This question is in two separate parts - the notation from part (i) is not used in part (ii).

   (i) It is useful to first show that $|Q|^2 = 52 = |A|^2$. Attempting to plot the hexagon will give only the points A and D on the hexagon, but the sketch will be helpful later in confirming the precise points found using algebra are sensible.

   To find point B you should consider translating and rotating the hexagon in the question until so that it is centred at the origin and points A and D lie on the real axis. The points of the transformed hexagon are now the solutions of the equation $z^6 = (\sqrt{52})^6 = 52^3$. Find the six different solutions to this equation and label them A', B', C', D', E', F' in a clockwise ordering where A' lies on the positive part of the real axis.

   To transform this heaxagon into the one in the question, one must rotate the hexagon (recall that multiplication by a unit length complex number gives a rotation of $\mathbb{C}$) and then translate the origin by Q.

   (ii) This question is straightforward. Rewrite the complex number in the question in the form $x + iy$ and find the condition satisfied when $y = 0$.