The King's Factor

This file contains specific solutions to parts of some questions set in Year 12 tKF sessions, so that you can check that you are on the right track. Some questions, for instance those where the result is given and a proof is required, do not have solutions given here.

These brief solutions are NOT model answers and do not indicate how a full solution should be presented

Updated February 20, 2014

1. Warm up

   (a) The curve crosses the y axis at $(0, -100)$.

   (b) The gradient of the curve is zero at $(2, -168)$ and $(-5, 175)$.

   (c) -

   (d) Since $\frac{\mathrm{d}y}{\mathrm{d}x} = 6(x-2)(x+5)$, the gradient of the curve is positive when $x < -5$ and when $x > 2$ while it is negative when $-5 < x < 2$. This means that $(2, -168)$ is a local minimum and $(-5, 175)$ a local maximum.

   (e) As $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to -\infty$.

   (f) This can now be done using the results above.

2. [2010 Oxford Admission Test question 1B] (Multiple Choice)

   The required sum is the sum of $1 + 2 + 4 + \cdots + 2^{n-1}$ and $1 + \frac{1}{2} + \frac{1}{4} \cdots + \frac{1}{2^{n-1}}$.

   This sum is $2^n - 1 + 2 - 2^n$ and so (a) is correct.

3. [2009 Oxford Admission Test question 1F] (Multiple Choice)

   The curve

   $$y = 3x^4 - 16x^3 + 18x^2 + k = 0$$

   has local minima at $(0, k)$ and $(3, k-27)$ and a local maximum at $(1, k+5)$.

   Also as $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to +\infty$.

   (This information makes a sketch straightforward.)

   For the equation to have four distinct solutions, the x axis must cross the graph at four points.

   This can only happen if the x-axis lies below the local maximum and above the greater of the two local minima.

   This will occur if $k + 5 > 0$ and $k < 0$. Hence the solution is (d).

4. [2009 Specimen paper 1 Oxford Admission Test question 1B] (Multiple Choice)

   (a)

5. [2005 STEP I question 2]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}\,.$$

   The equation of the tangent at P is $y = \frac{1}{p}x + p$ and the equation of the tangent at Q is $y = \frac{1}{q}x + q$.

   These meet where $x = pq$ and $y = p + q$, and so R is $(pq, p + q)$.

   The normals meet at $(p^2 + pq + q^2 + 2, p + q)$.

   If P and Q are such that $(1, 0)$ lies on the line PQ then $pq = -1$. Hence S has coordinates $(p^2 + q^2 + 1, p + q)$.

   We know that PR and PS are perpendicular, and that QR and QS are perpendicular.

   To show that $PQRS$ is a rectangle it is sufficient to show that PR and QR are perpendicular. This follows since these two lines have gradients $\frac{1}{p}$ and $\frac{1}{q}$ respectively and $pq = -1$.

6. Warm up
   (a) 92
   (b) 19

7. [2008 Oxford Admission Test question 1G] (Multiple Choice)
   The sum
   $$S(1) + S(2) + S(3) + \cdots + S(99)$$
   is equal to the sum of the numbers from 1 to 9 multiplied by 20.
   The answer is 900 and so (c) is correct.

8. [2008 Oxford Admission Test question 1D] (Multiple Choice)
   The remainder is
   $$1 + 3 + 5 + 7 + \cdots + 99 = 2500$$
   and so the answer is (b).

9. [2008 AEA question 2]
   (a) If P has coordinates (x, y)
   $$y = 2x + 5 \quad \text{and} \quad (x+1)(x+2)2 = xy\,.$$
   Hence $P = (-4, -3)$.
   (b) Separate variables to get
   $$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{(x+1)(x+2)}$$
   giving $y = k\dfrac{(x+2)^2}{x+1}$.
   Since $(-4, -3)$ lies on C the value of k is $\frac{9}{4}$.