Efficient Enumeration of Distinct Factors Using Package Representations

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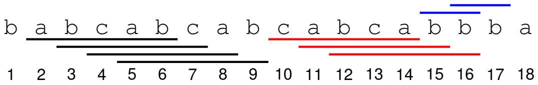

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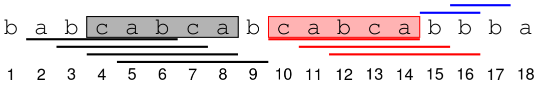

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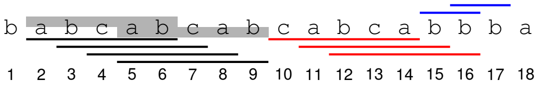

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Preliminaries

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E.g. *abcabcab* is periodic with period 3.

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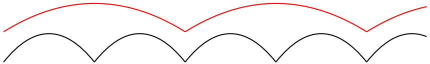

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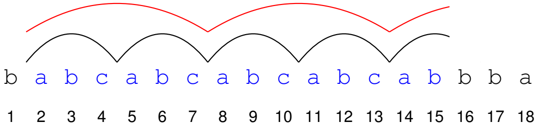

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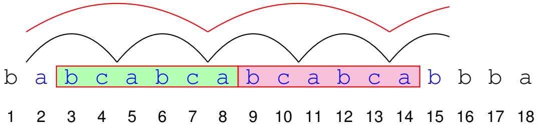

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Theorem

[Fraenkel-Simpson, J. Comb. Theory A 1996; Gusfield-Stoye, JCSS 2014]

A string of length n has $\mathcal{O}(n)$ distinct squares and they can be computed in $\mathcal{O}(n)$ time.

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A string of length n has $\mathcal{O}(n)$ runs and they can be computed in $\mathcal{O}(n)$ time.

In particular, an algorithm of [Crochemore et al., TCS 2014] extracts the distinct squares of a string from its runs in $\mathcal{O}(n)$ time.

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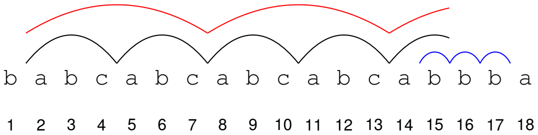

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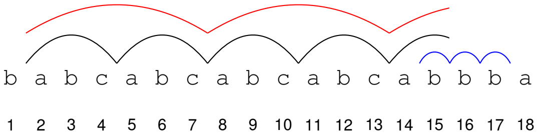

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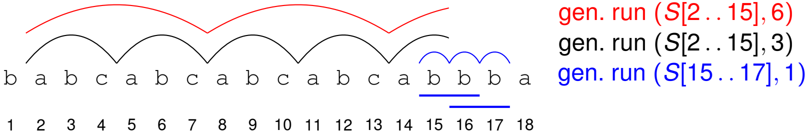

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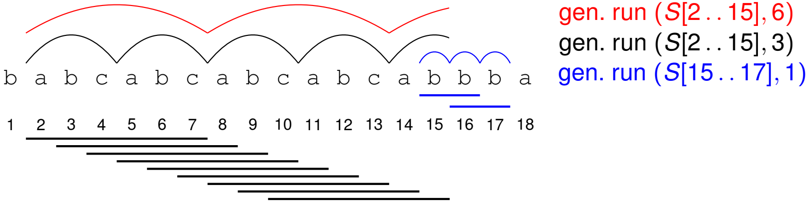

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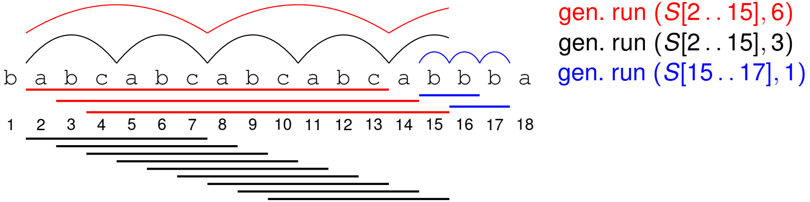

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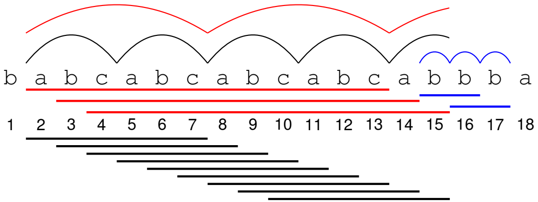

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Remark

This is related to computing the *subword complexity* of S.

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Observation

If $\mathcal{F}$ is special,

$$\mathsf{Factors}(\mathcal{F}) = \bigcup_{(i,\ell,k)\,\in\mathcal{F}} \{S[j\,..\,j+\ell) \;:\; j \in [i, i+k] \cap \mathsf{Smaller}_\ell\}.$$

Reporting in the Special Case

Algorithm 1: High-level structure of the algorithm.

$U := [1, n]$; $\mathcal{P} := \emptyset$

for $\ell := n$ **down to** 1 **do**

  $U := U \setminus \{j \; : \; LPF[j] = \ell\}$; // $U = \text{Smaller}_\ell$

  foreach $(i, \ell, k) \in \mathcal{F}$ **do**

    foreach $j \in [i, i+k] \cap U$ **do**

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We show an implementation of this idea in $\mathcal{O}(n + m + |\text{output}|)$, using the Union-Find data structure of [Gabow-Tarjan, JCSS 1985].

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We obtain an $\mathcal{O}(n+m)$-time algorithm by showing how to optimally answer these queries.

Counting in the Special Case II

Maintain array $A[1\,..\,n]$ such that during the ith phase:

$$A[\ell] = \begin{cases} i - \mathrm{Smaller}_\ell[i] & \text{if } \ell > LPF[i], \\ \mathrm{Smaller}_\ell[i] & \text{if } \ell \le LPF[i]. \end{cases}$$

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Maintain array $A[1\,..\,n]$ such that during the ith phase:

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For counting distinct k-antipowers, we improve over the $\mathcal{O}(nk^4 \log n \log k)$-time algorithm of [Kociumaka et al., arxiv].

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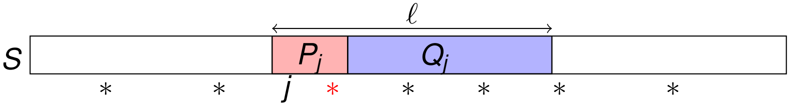

Idea: Assign each factor with $\ell \in [3\tau, 9\tau)$ to its first τ-synchroniser.

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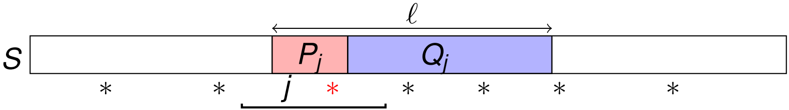

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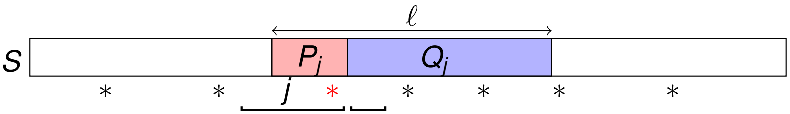

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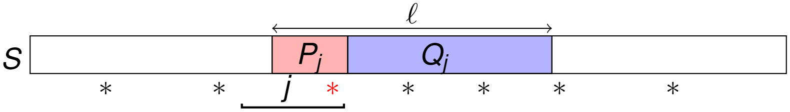

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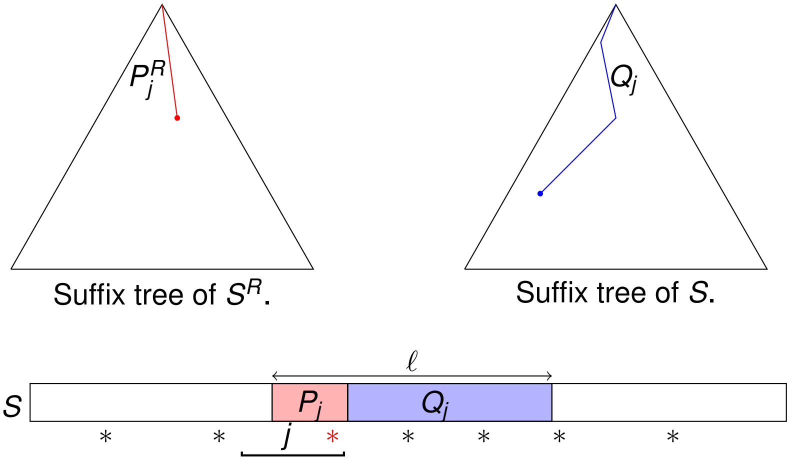

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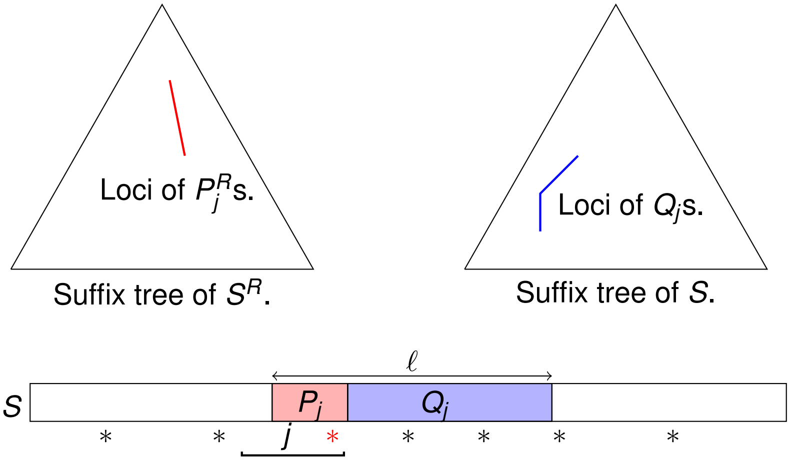

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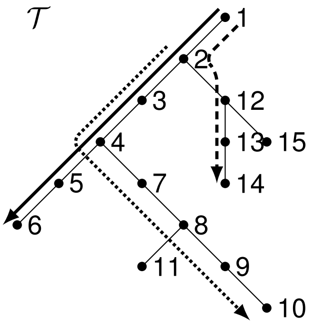

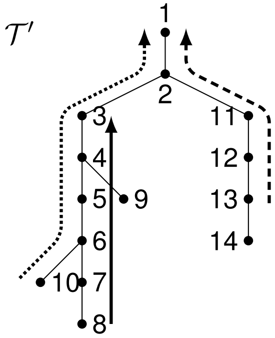

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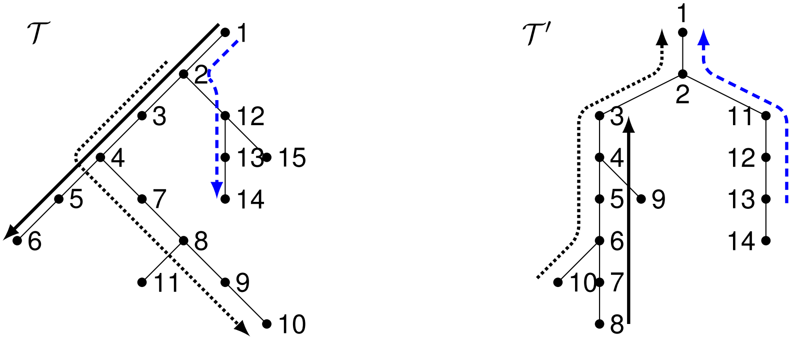

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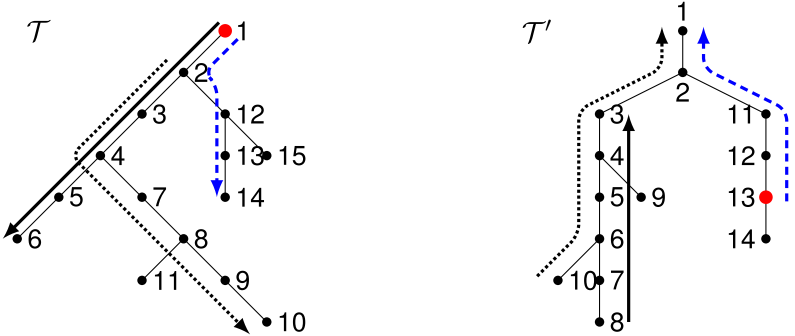

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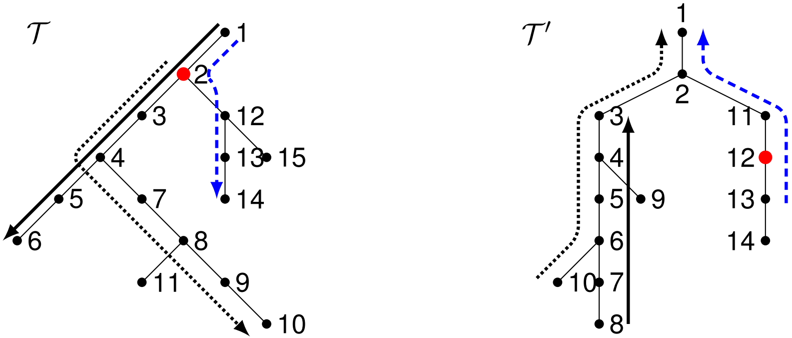

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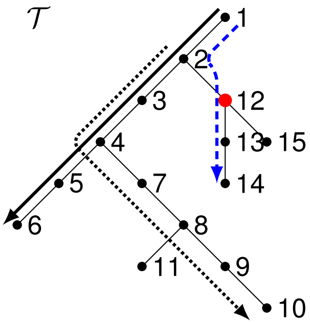

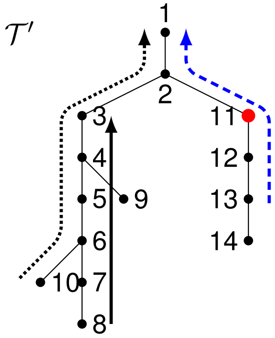

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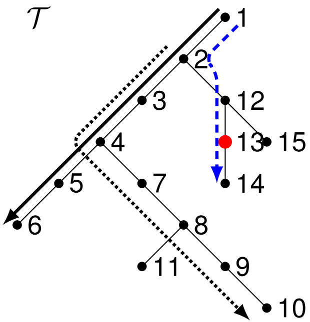

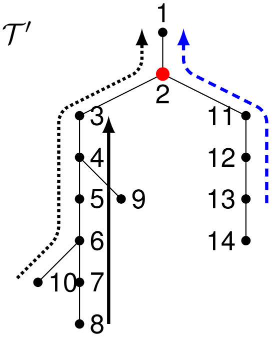

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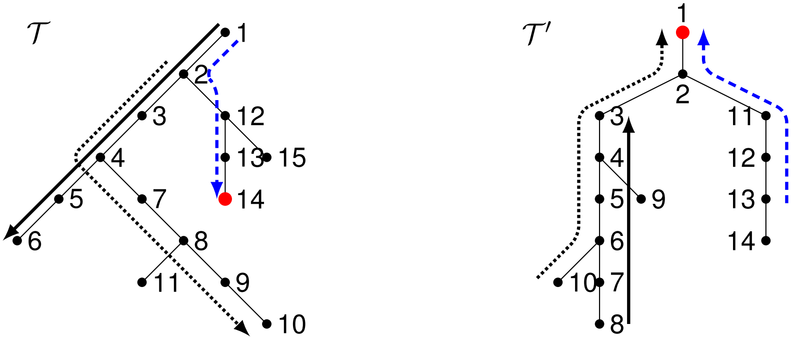

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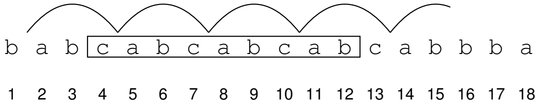

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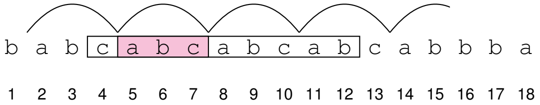

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b a b c a b c a b c a b c a b b b a

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

run $(S[2\,..\,15], 3)$

The General Case: Periodicity

We replace $\mathcal{F}$ by two sets of packages:

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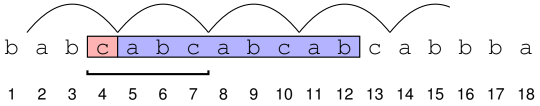

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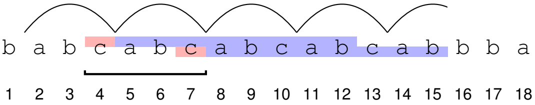

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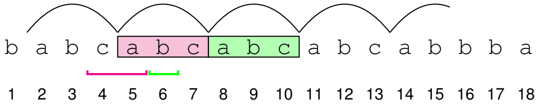

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We ensure that each package represents factors with the same period. Each package yields at most two pairs of paths.

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For a special package representation $\mathcal{F}$ consisting of m packages and a string of length n we can compute:

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Thank you for your attention!

Questions?