Single-Source Shortest Paths and Strong Connectivity in Dynamic Planar Graphs

Panagiotis Charalampopoulos[1,2] Adam Karczmarz[2]

[1]King's College London, UK

[2]University of Warsaw, Poland

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Dynamic Shortest Paths

Problem

Maintain a data structure over a graph G that supports:

- updates to the graph (e.g. edge insertions/deletions),
- shortest paths/distance queries.

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Unless explicitly stated otherwise, we consider directed weighted graphs.

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For sparse graphs (i.e. $m = \mathcal{O}(n)$), nothing better than recomputing from scratch is known for either of the variants.

Workarounds: settle for approximate answers, study more structured graph classes.

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We can maintain an n-vertex planar graph G under:

- edge insertions,
- edge deletions, and
- changes of the source s

in $\tilde{\mathcal{O}}(n^{4/5})$ worst-case time per update so that $\text{dist}_G(s, v)$ for any $v \in V(G)$ can be computed in $\mathcal{O}(\log^2 n)$ time.

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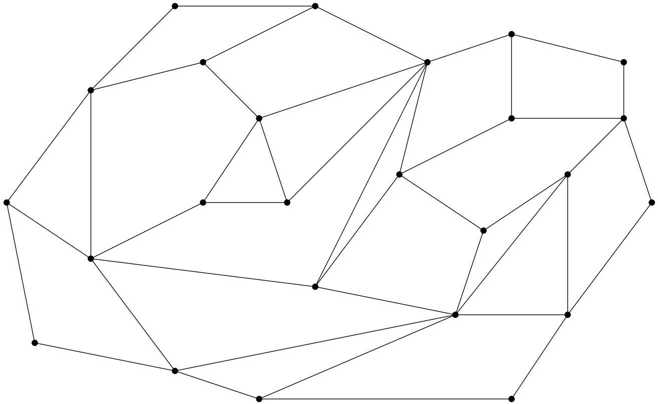
Our approach, combined with a few more ingredients, also yields a fully dynamic strong connectivity data structure with the same complexities.

Cycle Separators

Miller [JCSS'86]

There always exists a Jordan curve separator of size $\mathcal{O}(\sqrt{n})$ such that there are at most $\frac{2}{3}n$ vertices on its inside/outside.



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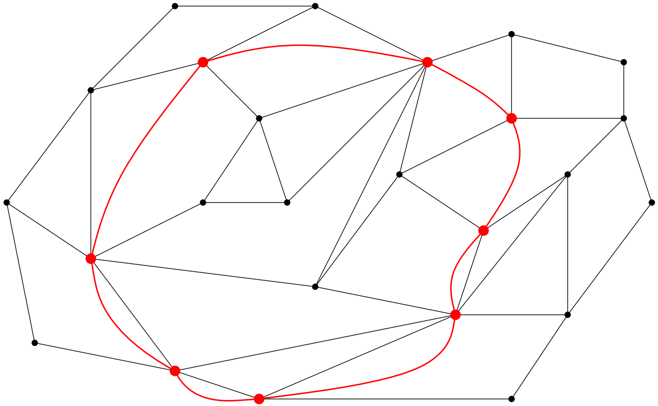
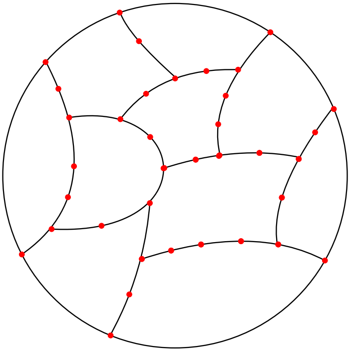


r-divisions

For $r \in [1, n]$, a decomposition of the graph into:

- $\mathcal{O}(n/r)$ pieces;
- each piece has $\mathcal{O}(r)$ vertices;
- each piece has $\mathcal{O}(\sqrt{r})$ boundary vertices (vertices incident to edges in other pieces).



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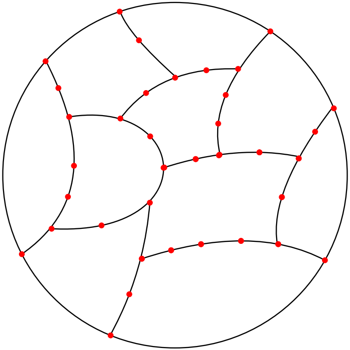

We denote the boundary of a piece P by ∂P and assume that all such vertices lie on a single face of P.

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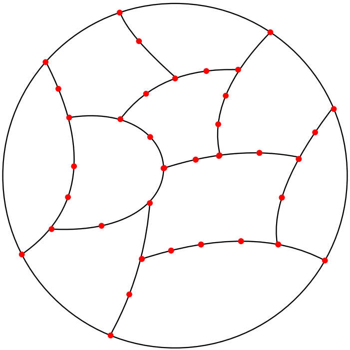

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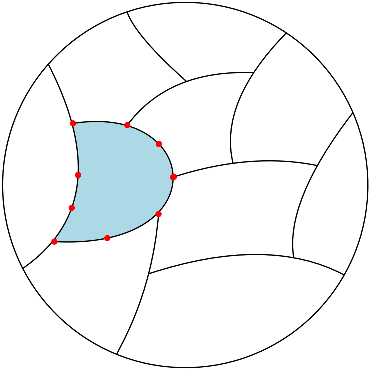
An r-division can be maintained in $\mathcal{O}(r)$ worst-case time per update with $\mathcal{O}(1)$ pieces changing. [Klein & Subramanian; WADS 1993]

Multiple Source Shortest Paths

MSSP [Klein; SODA 2005]

In nearly-linear time (in the size of the graph), we can construct a data structure that can report in logarithmic time the distance between any vertex u on the infinite face and any other vertex v of the graph.



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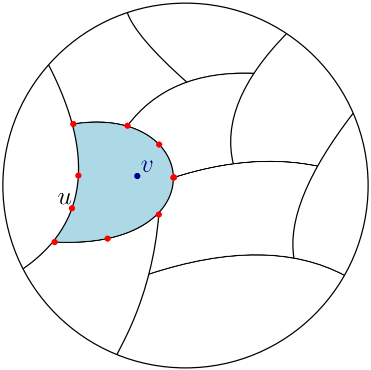

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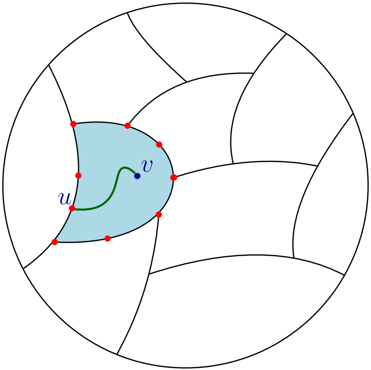
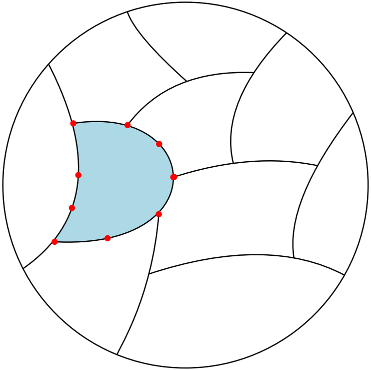


Dense Distance Graph (DDG)

The distance matrix capturing pairwise distances between vertices of a set ∂H of vertices lying on a single face of a plane graph H can be computed in $\tilde{O}(|H| + |\partial H|^2)$ time using MSSP.



FR-Dijkstra

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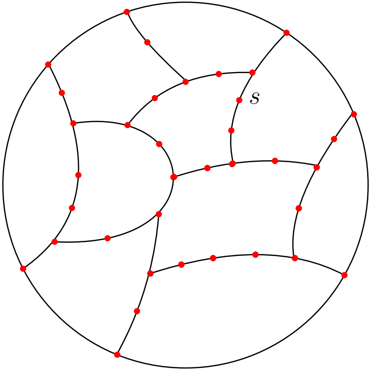
We can compute single-source shortest paths in a collection of DDGs with N vertices in total (with multiplicities) in $\tilde{O}(N)$ time.



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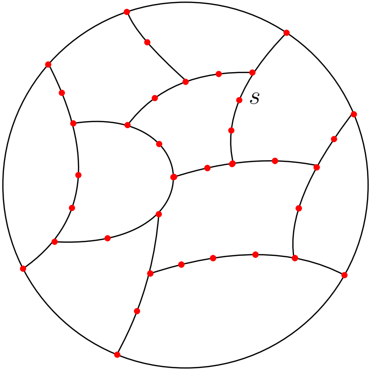

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- Maintain an r-division. $\mathcal{O}(r)$

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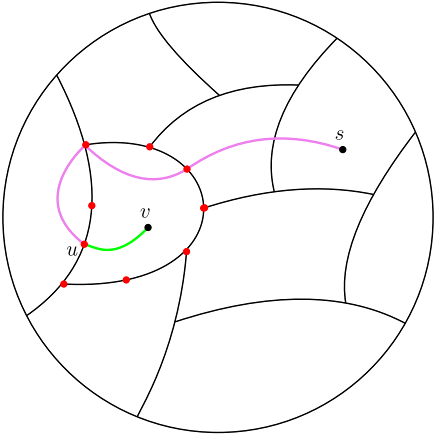
Balance for update time: $r = n^{2/3}$.

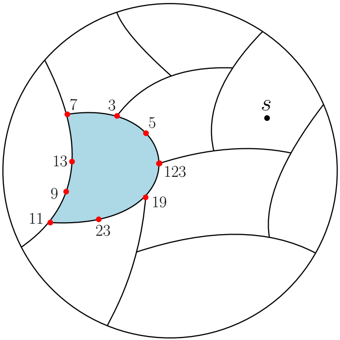
Point Location



Instead of trying all possible $\mathcal{O}(\sqrt{r})$ candidate boundary vertices, we want to compute the last boundary vertex u visited by the shortest path in $\tilde{\mathcal{O}}(1)$ time.

Point Location

- The $s \to \partial P$ distances define a set of additive weights for ∂P.



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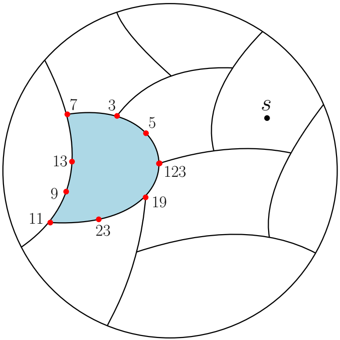

Point Location via Voronoi Diagrams

Given a set of additive weights for ∂P, there exists an $\tilde{\mathcal{O}}(\sqrt{r})$-sized data structure that given access to an MSSP data structure for P with sources ∂P answers point location queries in $\mathcal{O}(\log^2 n)$ time.

[Gawrychowski et al.; SODA'18]

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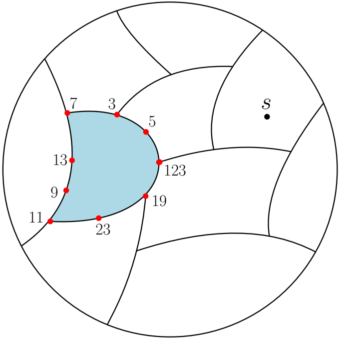

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[Gawrychowski et al.; SODA'18, Charalampopoulos et al.; STOC 2019]

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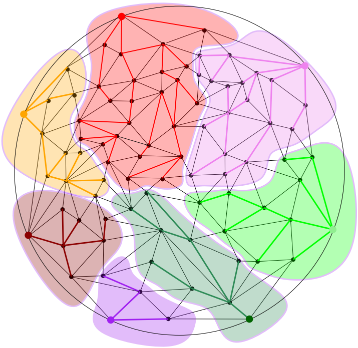

The Voronoi cell of each site consists of all vertices closer to it with respect to the additive distances.

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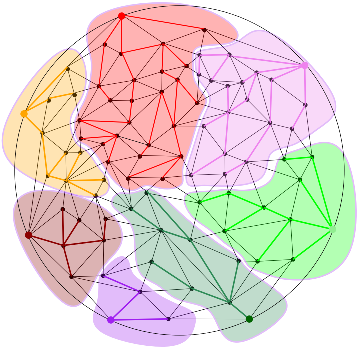

A point location query returns the Voronoi cell containing a queried vertex v.

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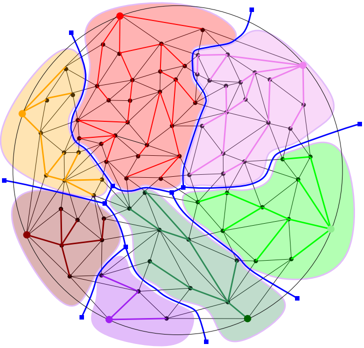

Because all sites are adjacent to one face, the diagram can be described by a tree on $\mathcal{O}(|\partial P|) = \mathcal{O}(\sqrt{r})$ vertices.

$\tilde{\mathcal{O}}(n^{4/5})$ update time, $\mathcal{O}(\log^2 n)$ query time

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- Maintain an r-division. $\mathcal{O}(r)$ time.
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Query:

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- *Construct a point location data structure for each piece P with $s \to \partial P$ distances as additive weights.* $\tilde{\mathcal{O}}(n/r \cdot r^{3/4}) = \tilde{\mathcal{O}}(n/r^{1/4})$

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$\tilde{\mathcal{O}}(n^{4/5})$ update time, $\mathcal{O}(\log^2 n)$ query time

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Balance: $r = n^{4/5}$.

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Plane Graphs

The dynamic plane transitive closure data structure of [Diks-Sankowski; ESA 2007] yields $\tilde{\mathcal{O}}(n^{1/2})$ update and query time.

Reachability Certificates

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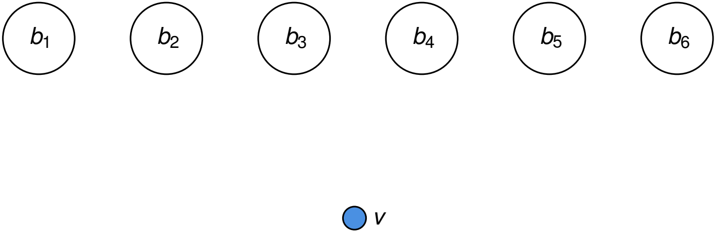

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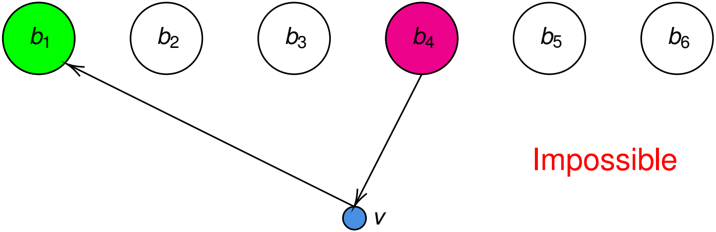

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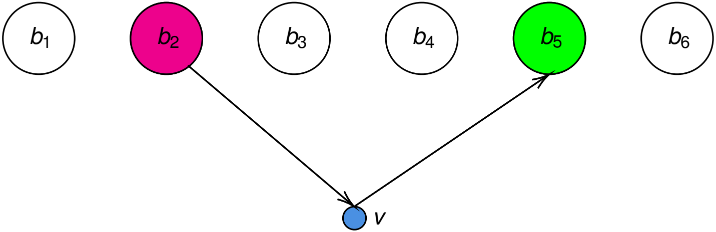

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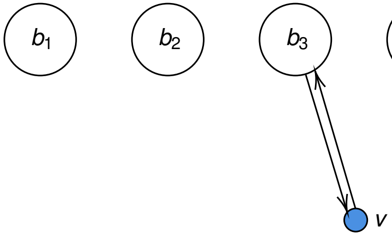

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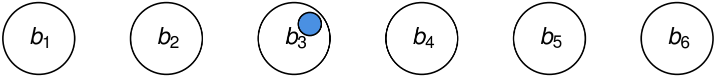

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Can more problems benefit?

Thank you for your attention!

Questions?