ECE 673-Random signal analysis I
Test 2 - Nov. 29, 2006

Please write legibly and provide detailed answers.

You are given an IID process $U[n]$ with probability mass function (PMF)

$$p_U[u] = P[U[n] = u] = \begin{cases} 1/2 & u = 1/2 \\ 1/2 & u = 2 \end{cases}.$$

(i) Consider the process $X[n] = \log_2 U[n]$. Is this process IID (and therefore stationary)? Find the PMF of $X[n]$ ($p_X[x] = P[X[n] = x]$), the average $E[X[n]]$ and variance $var(X[n])$.

Sol: Yes, the process is IID since applying a transformation (here the logarithm) to each $U[n]$ does not modify the statistical dependence among the variables of the process. Moreover, the range of $X[n]$ is $\mathcal{S}_X = \{-1, 1\}$ and the PMF reads

$$p_X[x] = \begin{cases} 1/2 & x = -1 \\ 1/2 & x = 1 \end{cases}.$$

Average and variance are as follows:

$$\begin{aligned} E[X[n]] &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0 \\ var(X[n]) &= E[X[n]^2] - E[X[n]] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1)^2 = 1. \end{aligned}$$

(ii) Consider now the process

$$Y[n] = X[n] - X[n-2]$$

Is $Y[n]$ WSS? In order to address this question, evaluate mean sequence $\mu_Y[n]$ and correlation sequence $c_Y[n, n+k]$. [notice that from the previous point $E[X[n]] = 0$ and $var(X[n]) = 1$].

Sol.: Mean sequence:

$$\mu_Y[n] = E[Y[n]] = E[X[n]] - E[X[n-2]] = 0.$$

Covariance sequence:

$$\begin{aligned} c_Y[n, n+k] &= E[Y[n]Y[n+k]] - E[Y[n]]E[Y[n+k]] = \\ &= E[Y[n]Y[n+k]] = E[(X[n] - X[n-2])(X[n+k] - X[n-2+k])] = \\ &= E[X[n]X[n+k]] + E[X[n-2]X[n-2+k]] + \\ &\quad -E[X[n]X[n-2+k]] - E[X[n-2]X[n+k]] \end{aligned}$$

Evaluating the previous expression for different values of k, we easily get

$$c_Y(n, n+k) = 2\delta[k] - \delta[k-2] - \delta[k+2] = c_Y[k] = r_Y[k]. \qquad (1)$$

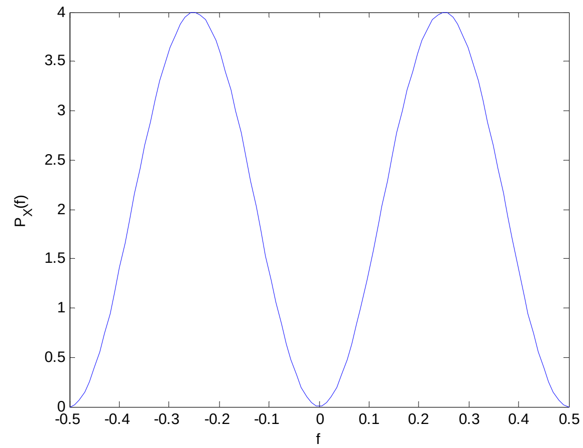

Figure 1:

From the above calculations, it follows that $Y[n]$ is WSS.

(iii) Evaluate and plot the power spectral density of $Y[n]$, $P_Y(f)$.

Sol.: In order to calculate the power spectral density $P_Y(f)$, we need to evaluate the discrete Fourier transform of the correlation function $r_Y[k]$ as

$$P_Y(f) = \sum_k r_Y[k] \cdot e^{-j2\pi fk} = 2(1 - \cos(4\pi f)).$$

See figure for plot.

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(iv) Evaluate the best linear predictor of $Y[n+1]$ given $Y[n]$ and the corresponding error.

Sol.: The correlation between $Y[n]$ and $Y[n+1]$ is zero from the answer to point (ii). Therefore, the best linear predictor of $Y[n+1]$ given $Y[n]$ is

$$\hat{Y}[n+1] = E[Y[n+1]] = 0,$$

and the corresponding mean square error is

$$mse = E[(\hat{Y}[n+1] - Y[n+1])^2] = var(Y[n+1]) = 2.$$

(v) Repeat the exercise above for the prediction of $Y[n+2]$ given $Y[n]$.

Sol.: The correlation between $Y[n]$ and $Y[n+2]$ is

$$cov(Y[n], X[Y+2]) = c_Y[2] = r_Y[2] = -1,$$

and the correlation coefficient reads

$$\rho[2] = \frac{cov(Y[n], Y[n+2])}{\sqrt{var(Y[n])var(Y[n+2)}} = \frac{r_Y[2]}{r_Y[0]} = -\frac{1}{2}.$$

Therefore, linear prediction is expected to be fairly effective. The predictor is

$$\begin{aligned}\hat{Y}[n+2] &= E[Y[n+2]] + \frac{cov(Y[n], Y[n+2])}{var(Y[n])}(Y[n] - E[Y[n]]) = \\ &= \frac{r_Y[2]}{r_Y[0]} Y[n] = -\frac{1}{2} Y[n]\end{aligned}$$

and the corresponding mean square error is

$$\begin{aligned}mse &= E[(\hat{Y}[n+2] - Y[n+2])^2] = var(Y[n+2])(1 - \rho[2]^2) =, \\ &= 2 \cdot (1 - 1/4) = 1.5.\end{aligned}$$

Linear prediction based on the knowledge of $Y[n]$ has reduced the mse from $var(Y[n+2]) = 2$ to 1.5.