ECE 673-Random signal analysis I
Test 1 - Oct. 4, 2006

Please write legibly and provide detailed answers.

We are interested in studying the connectivity of M towns along a country road (see figure, where $M = 4$). Because of unreliable weather conditions, the road between one town and the following is accessible only a fraction of the time.

1. For a generic M, define a reasonable sample space to study the problem at hand.
Sol.: Let us model each road as a different subexperiment and denote with "0" the probability of a road being closed and "1" open. The sample space then reads

$$\mathcal{S} = \{(z_1, z_2, ..., z_{M-1}) : z_i \in \{0, 1\}\}.$$

2. Assume that the connectivity on a road does not depend on the availability of any other road. Moreover, consider that the probability of a road being open is 0.7 and there are $M = 4$ towns. What is the probability that the first two towns (1 and 2) are connected, and so are 3 and 4, but there is no way of going from the the first pair of towns to the last?
Sol.: The requested probability reads

$$P[(1, 0, 1)] = P[1] \cdot (1 - P[1]) \cdot P[1] = 0.7 \cdot 0.3 \cdot 0.7 = 0.147.$$

3. Now assume that the quality of a road depends on the preceding (e.g., the connectivity between 2 and 3 depends on the connectivity between 1 and 2). In particular, consider that the probability of a road being available is 0.9 if the preceding is available, while is 0.2 if the preceding is not. Finally, assume that the first road (between 1 and 2) is open with probability 0.7. For $M = 4$, find the probability that the first two towns (1 and 2) are connected, and so are 3 and 4, but there is no way of going from the the first pair of towns to the last. Compare your result with question 2.
Sol.: Let us denote as $P[A_n|A_{n-1}]$ as the probability of some event A_n relative to the nth link given event A_{n-1} relative to the $n - 1$ link. From the problem statement, we have

$$\begin{aligned} P[1|1] &= 0.9 \\ P[1|0] &= 0.2 \end{aligned}$$

The probability at hand then reads

$$P[(1, 0, 1)] = P[1] \cdot (1 - P[1|1]) \cdot P[1|0] = 0.7 \cdot 0.1 \cdot 0.2 = 0.014$$

4. Consider a long road with many towns ($M = 10001$). Connection qualities are independent as in question 2. The weather in the region is so bad that the probability of a road being accessible is 0.001. What is the probability that five roads are open (you can use some approximation if you prefer)? What is the average number of accessible roads?
Sol.: The real probability density function of $X = \sum_{k=1}^{M-1} z_k$ is $bin(M-1, 0.001)$, i.e., $X \sim bin(10000, 0.001)$. Since the number of trials is large and the probability of "hit" is small, a good approximation is $X \sim Pois(10000 \cdot 0.001 = 10)$. The probability requested is

$$P[X = 5] = \exp(-10)\frac{10^5}{5!} = 0.038.$$

The average of X is $E[X] = 10$.

5. Say that we have $M = 3$ towns with independent links, and that the probability of a road being accessible is 0.4. Write the *equation and plot* the probability mass function and cumulative distribution function of a random variable Y that counts the number of available links.
6. Write a MATLAB code that generates the random variable Y defined at the previous question.

Sol.: We have $Y = \sum_{k=1}^{3} z_k \sim bin(2, 0.4)$ (see textbook and notes for plots and equations of PMF and CDF). A MATLAB code that generates this random variable, using the inverse CDF transform, is as follows:

```
u=rand(1);
 if (u<=0.36)
   x=0;
 elseif (u>0.36)&(u<=0.84)
    x=1;
  elseif (u>0.84)
   x=2;
 end %if
```