ECE 788 - An Introduction to Quantum Communication Theory

Midterm

1. (**1 point**) Consider two boxes, one emitting quantum states $|x+\rangle$ and $|x-\rangle$, and the other states $|z+\rangle$ and $|z-\rangle$, with equal probability. Is it possible to devise a basic measurement that is able to distinguish the two boxes? Please motivate your response.

2. (**2 points**) Consider the cascade of two filters, where the first filter makes a measurement of a photon with the basis $\{\cos(\pi/8)\,|z+\rangle + \sin(\pi/8)\,|z-\rangle, -\sin(\pi/8)\,|z+\rangle + \cos(\pi/8)\,|z-\rangle\}$ and passes only the component along the first vector, and the second filter uses the basis $\{|x+\rangle, |x-\rangle\}$ and passes only the second component.
a. Calculate the probability that the photon passes through the two filters if its initial state is $|z+\rangle$.
b. Suppose now that the first filter operates using the basis $\{|x+\rangle, |x-\rangle\}$ and passes the first component. Calculate the probability of the photon passing the two filters and propose to introduce an intermediate filter that increases this probability.

3. (**3 points**) We have the two matrices

$$S_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$S_2 = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$$

a. Show that they are observables.
b. Calculate expectation and variance of the two observables for state $1/\sqrt{2}(|0\rangle + |1\rangle)$ (Hint: For an observable, we have $S^2 = S \cdot S$).
c. Compare the results above with the inequality provided by the uncertainty principle.

4. (**1 point**) Prove that for any positive operator P and normalized vector $|\psi\rangle$, we have

$$\langle\psi| P |\psi\rangle \leq Tr(P).$$

5. (**1 point**) Write the operator $|0\rangle\langle 1|$ as the sum of a Hermitian and an anti-Hermitian operator.

6. (**1 point**) Prove that if U is unitary (i.e., $U^\dagger U = 1$), then we also have $UU^\dagger = 1$. As a preliminary step, prove that, if $\{|k\rangle\}$ is an orthonormal basis, then $\{U|k\rangle\}$ is also an orthonormal basis.

7. (**1 point**) Alice sends quantum states $|x+\rangle$ and $|x-\rangle$ with equal probability and Bob makes measurements using the basis $\{|x+\rangle, |x-\rangle\}$. Eve intercepts the quantum states and makes measurements choosing bases $\{|x+\rangle, |x-\rangle\}$ or $\{|z+\rangle, |z-\rangle\}$ with equal probability. What is the probability that the state measured by Bob is different from that sent by Alice?

8. (**1 point**) Suppose that $|\psi\rangle$ is an eigenvector of both observables A and B (no other assumption is made on A and B). Show that $|\psi\rangle$ is also an eigenvector of $[A, B]$ with eigenvalue 0. What can we say about the anti-commutator $\{A, B\}$? What does the uncertainty relation say about state $|\psi\rangle$?