**ECE 755 - Digital communications**
**Midterm**

Please provide clear and complete answers (and write legibly!)

**PART I: Questions -**

**Q1.** (1 point) Consider a modulator that takes as input $b$ bits every $\tau$ seconds and produces as output three real symbols $a_1, a_2, a_3$. Symbols $a_1, a_2, a_3$ are then transmitted via an orthonormal basis of functions $\phi_i(t)$ over a bandwidth $B$ and time $\tau$. What is the minimum bandwidth $B$ needed for this system (assuming $\tau$ sufficiently large)? What is the maximum spectral efficiency of this system? What is the number of bits/ 2-dim?

*Sol.*: From the Laudau-Pollack theorem, we have

$$B \geq \frac{3}{2\tau}.$$

The spectral efficiency (bits/ sec/ Hz) is

$$\nu = \frac{b}{\tau} \cdot \frac{1}{B} \leq \frac{2}{3}b,$$

while the number of bits/ 2-dim is $2b/3$.

**Q2.** (1 point) Given two possible sequences $\mathbf{a}' = [1\ 1\ 1]$ and $\mathbf{a}'' = [1\ -1\ -1]$ observed over an AWGN channel with noise power per dimension $\sigma^2 = N_o/2 = 0.3$, evaluate the likelihoods for $\mathbf{a}'$ and $\mathbf{a}''$ given the received signal samples $\mathbf{y} = [1.3\ -0.6\ 0.8]$ (i.e., $f(\mathbf{y}|\mathbf{a})$). Then, assume that the a priori probabilities for the sequences are $p(\mathbf{a}') = 0.2$ and $p(\mathbf{a}'') = 0.8$, and find the a posteriori probabilities of $\mathbf{a}'$ and $\mathbf{a}''$ given the same $\mathbf{y}$ (you can neglect inessential multiplicative constants), and the their ratio. What is the output of the MLSD and of the MAPSD?

*Sol.*: The likelihoods are as follows:

$$\begin{aligned} f(\mathbf{y}|\mathbf{a}') &= f(y_0|a_0 = 1)f(y_1|a_1 = 1)f(y_2|a_2 = 1) = \\ &= \frac{1}{(2\pi \cdot 0.3)^{3/2}} \exp\left(-\frac{(1.3-1)^2 + (-0.6-1)^2 + (0.8-1)^2}{2 \cdot 0.3}\right) = 0.0044 \\ f(\mathbf{y}|\mathbf{a}'') &= \frac{1}{(2\pi \cdot 0.3)^{3/2}} \exp\left(-\frac{(1.3-1)^2 + (-0.6+1)^2 + (0.8+1)^2}{2 \cdot 0.3}\right) = 0.0012 \end{aligned}$$

and the log-likelihood ratio is:

$$\log \frac{f(\mathbf{y}|\mathbf{a}')}{f(\mathbf{y}|\mathbf{a}'')} = \log \frac{0.0044}{0.0012} = 1.3.$$

A posteriori probabilities:

$$\begin{aligned} p(\mathbf{a}'|\mathbf{y}) &= \frac{p(\mathbf{a}') \cdot f(\mathbf{y}|\mathbf{a}')}{f(\mathbf{y})} \propto p(\mathbf{a}') \cdot f(\mathbf{y}|\mathbf{a}') = 0.2 \cdot 0.0044 = 8.8 \cdot 10^{-4} \\ p(\mathbf{a}''|\mathbf{y}) &= \frac{p(\mathbf{a}'') \cdot f(\mathbf{y}|\mathbf{a}'')}{f(\mathbf{y})} \propto p(\mathbf{a}'') \cdot f(\mathbf{y}|\mathbf{a}'') = 0.8 \cdot 0.0012 = 9.6 \cdot 10^{-4} \end{aligned}$$

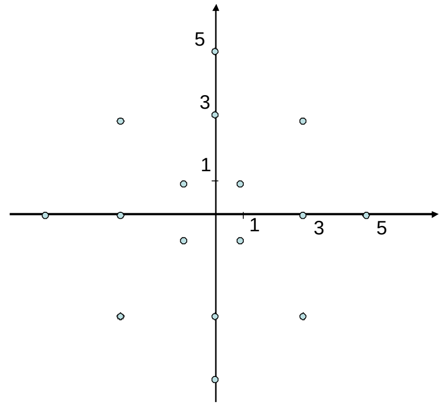


Figure 1:

with log-likelihood ratio:

$$\log \frac{p(\mathbf{a}'|\mathbf{y})}{p(\mathbf{a}''|\mathbf{y})} = \log \frac{8.8}{9.6} = -8.7 \times 10^{-2}.$$

Both decision are quite unreliable, but the MLSD decides for $\mathbf{a}'$ while the MAPSD for $\mathbf{a}''$.

**Q3.** (1 point) Consider the constellation of points $a \in \mathcal{A}$ in Fig. 1 used for transmission over a standard AWGN channel with power spectral density $N_0/2$.
**a.** Write a general formula for the probability of symbol error for the constellation at hand and equally likely transmitted symbols in terms of $f(\mathbf{y}|a)$, where $\mathbf{y}$ is the two-dimensional received vector. Explain why (and if) this calculation is complicated (what step would you not be able to carry out?).
**b.** From the general formula, write the steps that lead to the union bound approximation, and evaluate such an approximation.
**c.** For what values of $N_0$ you expect the approximation to be accurate? Why?

*Sol.*: **a.**

$$P_e = \frac{1}{16}\sum_{k=1}^{16} \Pr[\text{error}|A = a_k],$$

with

$$\Pr[\text{error}|A = a_k] = 1 - \int_{\text{decision region for } a_k} f(\mathbf{y}|A = a_k)d\mathbf{y},$$

with $f(\mathbf{y}|A = a_k)$ being the likelihood of the received signal, which is a two-dimensional multivariate Gaussian distribution. Since the decision regions here have complicated (i.e., non-separable or in other words non rectangular) shapes, such integration is not easy.

**b.**

$$
\begin{aligned}
P_e &= \frac{1}{16}\sum_{k=1}^{16} \Pr[\text{error}|A = a_k] \\
&= \frac{1}{16}\sum_{k=1}^{16}\sum_{i\neq k} \Pr[\hat{A} = a_i|A = a_k] \\
&\leq \frac{1}{16}\sum_{k=1}^{16}\sum_{i\neq k} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right) \\
&\simeq K_{\min,ave}\, Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right),
\end{aligned}
$$

where $d_{\min}$ is the minimum distance and $K_{\min,ave}$ the average number of neighbors at the minimum distance. Here we have $d_{\min} = 2$ and by a direct calculation

$$
K_{\min,ave} = \frac{1}{4}2 + \frac{1}{4}0 + \frac{1}{4}1 + \frac{1}{4}1 = 1.
$$

**c.** For small $N_0$, since we are neglecting terms that are small at high SNR.

**Q4**. (1 point) You are given a radio receiver with $M$ antennas that is connected to a PC for digital processing (i.e., this is a software-defined radio). You can design the radio frequency (analog) part of the receiver by using local oscillators at frequency $f_{LO}$, linear filters and *two* analog-to-digital converters (ADC) working each at $64Msamples/s$ (real samples). The ADCs connect the analog world to the PC. Assume that the signal of interest has a bandwidth $W$ centered around frequency $f_c = 1GHz$ and that there are no synchronization issues.
**a.** Suppose that the available local oscillators have $f_{LO} = 984MHz$. Propose a RF receiver (with the blocks described above) that enables alias-free ADC conversion. How large can $W$ be? How many antennas (i.e., received signals) can this receiver handle (given the available ADC converters)?
**b.** Assume now that $f_{LO} = 1GHz$ and respond to the same questions at the previous point.

*Sol.*: **a.** A reasonable structure is the following (heterodyne) receiver. The received signal is filtered to remove image frequencies and then multiplied by $\cos(2\pi f_{LO})$ and filtered again to remove high-frequency components. As a result, the central frequency is then shifted to $f_c - f_{LO} = 16MHz$. In order to avoid alias when sampling, the bandwidth $W$ has to satisfy $W \leq 32MHz$. Since this requires only one ADC, we can accommodate $M = 2$ antennas with this design.
**b.** In this case, the receiver may shift the central frequency to DC. The receiver is then the standard (homodyne) structure with two branches where the signal is multiplied in the first by $\sqrt{2}\cos(2\pi f_{LO})$ and in the second by $-\sqrt{2}\sin(2\pi f_{LO})$. Each branch, after low-pass filtering, is then fed to an ADC. The maximum bandwidth that can be tolerated without alias is now $W \leq 64MHz$. However, since we need two ADC, we can only accommodate $M = 1$ antenna with the available hardware.

**PART II: Problems -**

**P1.** Consider a wireless sensor powered by a battery with a residual energy $E_{battery} = 10J$. Assume that the sensor can transmit with powers $P_1 = -6dBm$, $P_2 = -3dBm$ or $P_3 = 0dBm$ and that the attenuation between transmitter and receiver is $\eta = 50dB$ (so that the received power is $P_i/\eta$). We neglect any other form of energy consumption at the sensor. The bandwidth used for transmission is $W = 1kHz$ around carrier frequency $f_c = 2.4GHz$ and the power spectral density of the additive white Gaussian noise is $N_0 = 10^{-11}W/Hz$.
**a.** Consider at first ideal coding (i.e., no constraints on complexity and delay). **a.1**) Calculate maximal achievable rates in bits/ sec for the three different powers. **a.2**) Then, calculate the sensor life-time in hours (i.e., how long will the sensor last if it transmits continuously?) and the maximum number of bits that the sensor can transmit in its life-time for the three powers. Is it better to transmit with a smaller or larger power if one wants to maximize the number of transmitted bits?
**b.** Following the point above, if you were free to choose whatever power and spectral efficiency you wish for the sensor, what would you choose to maximize the number of bits sent *reliably* during the life-time of the sensor? Calculate such maximum number of bits. (*Hint*: The minimum energy per bit required for reliable communications is...)
**c.** Assume now that transmission takes place via BPSK modulation with roll-off factor $\alpha = 0.2$, and that we are interested in obtaining a probability of bit error less than 0.3. Find the power among $P_1$, $P_2$ and $P_3$ that maximizes the number of bits sent in the sensor's life-time under the constraint above. What is such maximum number of bits?

*Sol.*: **a. a.1)**

$$\begin{aligned}
R_1 &= W\log_2\left(1+\frac{P_1/\eta}{WN_0}\right) = W\log_2\left(1+\frac{10^{-5}\cdot 1/4\cdot 10^{-3}}{10^3 10^{-11}}\right) = 0.322\ kbits/s \\
R_2 &= W\log_2(1+1/2) = 0.585\ kbits/s \\
R_3 &= W\log_2(1+1) = 1\ kbits/s
\end{aligned}$$

**a.2)** Life-times

$$\begin{aligned}
L_1 &= \frac{E_{battery}}{P_1} = 40000\,\text{sec} \simeq 11.1\ hours \\
L_2 &\simeq 5.6\text{ hours} \\
L_3 &\simeq 2.8\text{ hours}
\end{aligned}$$

and overall number of bits

$$\begin{aligned}
N_{b1} &= L_1R_1 \simeq 13\ Mbits \\
N_{b2} &\simeq 11.7\text{ Mbits} \\
N_{b3} &= 10\text{ Mbits}
\end{aligned}$$

**b.** We know that the minimum received energy ber bit received satisfies $E_{b\min,rx}/N_0 = -1.6dB = \log 2$, so that

$$E_{b\min,rx} = N_0\cdot\log 2 \simeq 0.7\cdot 10^{-11}J$$

and the corresponding transmitted energy per bit is

$$E_{b\min,tx} = \eta E_{b\min,rx} = 0.7\cdot 10^{-6}J.$$

The maximum number of bits is thus

$$N_{b\max} = E_{battery}/E_{b\min,tx} \simeq 1.43 \cdot 10^3 \ Gbits.$$

**c**. Since we are using BPSK, the number of bits sent every second equals

$$\frac{W}{(1+\alpha)} \cdot \rho = \frac{W}{(1+\alpha)} = 83.3 \text{ bits/sec},$$

so that the total number of bits is

$$\frac{W}{(1+\alpha)} \cdot L_i.$$

Clearly, choosing $P_1$ maximizes the life-time and thus the total number of bits. We need to check however that the probability of error constraint is satisfied

$$\begin{aligned} Q(\sqrt{\frac{2E}{N_0}}) &= Q(\sqrt{\frac{2P_1}{N_0 W}(1+\alpha)}) \\ &= Q(\sqrt{\frac{2 \cdot 1.2 \cdot 10^{-5}}{4 \cdot 10^{-8} 10^3}}) = Q(\sqrt{0.6}) \simeq 0.22 < 0.3. \end{aligned}$$

The number of bits is thus

$$\frac{W}{(1+\alpha)} \cdot L_1 = 83.3 \cdot 40000 = 333.2 \ kbits.$$

**P2.** Consider a PAM transmission with equivalent channel $h(t)$ being a rectangle of duration $\tau$ (i.e., from 0 to $\tau$) with unit energy ($h(t) = 1/\sqrt{\tau} \cdot rect(t/\tau)$, where $rect(t) = 1$ for $0 \leq t \leq 1$ and zero otherwise). The received signal is $Y(t) = \sum_k a_k h(t-kT) + N(t)$ with $1/T = 1Mbaud$ and the usual definitions. A BPSK constellation $\mathcal{A} = \{\pm c\}$ is used. Assume that $L = 3$ symbols are sent ($a_0$, $a_1$ and $a_2$) and that before ($k < 0$) and after ($k > 1$) the data sequence, symbols $a_k = -1$ are sent.
**a**. Assuming $\tau = 1\mu s$ **a.1**) Sketch the matched filter (MF) followed by a $T$-spaced sampler, and write the corresponding output signal $Y_k$; **a.2**) Write down the MLSD rule and find the detected symbols given the received samples $[y_0 \ y_1 \ y_1] = [0.1 - 0.7 \ 0.3]$.
**b**. Consider now the case where $\tau = 1.5\mu s$. **b.1**) Write the expression of the received signal $Y_k$ after matched filtering and $T$-spaced sampling; **b.2**) Find the Whitened Matched filter (WMF) and the MLSD rule.
**c.** For the case at point b., assume that $c = 1$ and that the received signal after the WMF is $[z_0 \ z_1 \ z_2 \ z_3] = [-1.6 \ 0.8 \ -0.9 \ -1.9]$. Find the detected sequence according to the MLSD using the Viterbi algorithm.

*Sol.*: **a. a.1**) For the block diagram, please see notes and textbook. At the output of the MF we have channel correlation function

$$\begin{aligned} \rho_h(k) &= h(t) * h(-t)|_{t=kT} \\ &= \delta_k, \end{aligned}$$

since $h(t) * h(-t)$ is a triangle with base corresponding to the interval $[-1\mu s, 1\mu s]$, while $T = 1\mu s$. It follows that the signal at the output of the MF and sampler is

$$Y_k = a_k + N_k',$$

with white noise with power spectral density $N_0$.
**a.2**) The MLSD rule is ($L = 3$ here)

$$\begin{aligned} &\min_{a \in \mathcal{A}^L} -\log f(\mathbf{y}|\mathbf{a}) \\ \Leftrightarrow \quad &\min_{a \in \mathcal{A}^L} \sum_{k=0}^{L-1} (y_k - a_k)^2, \end{aligned}$$

which can be solved symbol by symbol. For the given example, the detected symbols are $c$, $-c$ and $c$, respectively.
**b. b.1)** Here we have

$$\begin{aligned} \rho_h(k) &= h(t) * h(-t)|_{t=kT} \\ &= \delta_k + \frac{1}{3}\delta_{k-1} + \frac{1}{3}\delta_{k+1} \end{aligned}$$

and the folded spectrum is

$$\begin{aligned} S_h(z) &= 1 + \frac{1}{3}z^{-1} + \frac{1}{3}z \\ &= \gamma^2 \left(1 + dz^{-1}\right)(1 + dz), \end{aligned}$$

where it can be checked that, in order to have $|d| < 1$ as desired, we need $d = (3 - \sqrt{5})/2 \simeq 0.38$ and $\gamma^2 = 1/3d \simeq 0.87$. The received signal after sampling is

$$Y_k = a_k + \frac{1}{3}a_{k-1} + \frac{1}{3}a_{k+1} + N_k',$$

where the power spectral density of the noise is $S_{N'}(z) = N_0 S_h(z)$.
**b.2**) The whitening filter at the output of the sampler is

$$\frac{1}{\gamma^2 M^*(1/z^*)} \simeq \frac{1}{0.87 \cdot (1 + 0.38z)}.$$

The received signal at the output of the WMF is

$$Z_k = a_k + 0.38a_{k-1} + N_k'',$$

where the power spectral density of the noise is $N_0/\gamma^2$. The MLSD rule is

$$\min_{a \in \mathcal{A}^L} \sum_{k=0}^{L-1} (z_k - (a_k + 0.38a_{k-1}))^2.$$

The trellis is shown in Fig. 2. The Viterbi algorithm leads to decode (-1,1,-1).

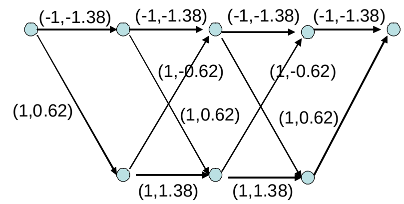


Figure 2:

**P.3.** (2 points) In partial response systems, intersymbol interference is purposedly created to improve the system performance. Consider a partial response system where OOK (on-off keying) information symbols $x_k \in \{0,1\}$, selected independently for $k = 0, 1, ..., L-1$, are prefiltered by $H(z) = 1 - cz^{-1}$ with $c$ a given (real) constant, producing as output a sequence $a_k$, which is transmitted via passband PAM as $s(t) = \sum_{k=0}^{L-1} a_k g(t - kT)$, where $g(t)$ is a standard squared root of Nyquist waveform with zero roll-off factor.
**a.** Find and *sketch* the power spectral density $S_s(f)$ of the transmitted signal $s(t)$.
**b.** The signal $s(t)$ can be equivalently seen as a regular PAM transmission with OOK symbols $x_k$, i.e., $s(t) = \sum_{k=0}^{L-1} x_k h(t - kT)$ and equivalent waveform $h(t)$. Find $h(t)$. Calculate and sketch the Fourier transform $|H(f)|^2$.
**c.** Is $h(t)$ a square root of Nyquist waveform? If not, for what values of $c$ can you find a filter $f(t)$ such that $f(t) * h(t)$ satisfies this condition?
**d.** Consider $L = 2$ and draw in the (two-dimensional) signal space the four possible received points $(a_0, a_1)$ (we neglect $a_2$) for the three cases $c = 0$, $c = -1$ and $c = 1$. Fixing the average constellation energy, what constellation(s) has (have) the best performance?
*Sol.*:
**a.**

$$
\begin{aligned}
S_s(f) &= \frac{1}{T} S_a(e^{j2\pi fT}) |G(f)|^2 \\
&= \begin{cases} \frac{1}{T} S_a(e^{j2\pi fT}) & \frac{1}{2T} - \leq f \leq \frac{1}{2T} \\ 0 & \text{elsewhere} \end{cases},
\end{aligned}
$$

with

$$
\begin{aligned}
S_a(e^{j2\pi fT}) &= E_x |1 - ce^{-j2\pi fT}|^2 \\
&= \frac{1}{2} \left(1 + c^2 - 2c\cos(2\pi fT)\right)
\end{aligned}
$$

**b.**

$$\begin{aligned} s(t) &= \sum_{k=0}^{L-1} a_k g(t-kT) \\ &= \sum_{k=0}^{L-1} (x_k - c x_{k-1}) g(t-kT) \\ &= \sum_{k=0}^{L-1} x_k (g(t-kT) - c g(t-(k-1)T)), \end{aligned}$$

so that

$$h(t) = g(t) - cg(t-T).$$

We have

$$\begin{aligned} |H(f)|^2 &= |G(f)|^2 |1 - ce^{-j2\pi fT}|^2 \\ &= \begin{cases} 1 + c^2 - 2c\cos(2\pi fT) & \frac{1}{2T} - \leq f \leq \frac{1}{2T} \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

**c.** No, since $\sum_k |H(f-k/T)|^2$ is clearly not constant. Filter $f(t)$ that satisfy the stated conditions can found as $F(f) = 1/H(f)$ in the band of interest $\frac{1}{2T} - \leq f \leq \frac{1}{2T}$, unless $H(f)$ has a zero in such bandwidth. This case happens only if $c = \pm 1$.

**d.** The two constellations for $c = 0$ is $((0,0),(0,1),(1,0),(1,1))$, for $c = -1$ is $((0,0),(0,1),(1,-1),(1,0))$ and for $c = 1$ is $((0,0),(0,1),(1,1),(1,2))$. It can be easily seen that with a constraint on the average energy the last two constellations will have a smallest minimum distance than the other two, so that $c = 0$ is to be preferred.