ECE 744

Midterm, Fall 2014

Please provide clear and complete answers by detailing your derivations.

1. (**2 points**) Consider the cone $K = \{x \in \mathbb{R}^n | x = Ay \text{ with } y \succeq 0\}$, with $A \in \mathbb{R}^{n\times k}$ and $y \in \mathbb{R}^k$.
a. Calculate the dual cone (Hint: use the definition).
b. Consider the case $n = 2$ and $k = 2$. Give conditions on A so that the cone K is proper.

2. (**1 point**) Consider a convex function $f:\ \mathbb{R} \rightarrow \mathbb{R}$ with domain $\mathbf{dom} f = \{x | a \leq x \leq b\}$ that is monotonically increasing. Is the inverse function $f^{-1}(x)$ convex/concave? Is it quasi-convex and/or quasi-concave? (Hint: Recall that $f^{-1}(f(x)) = x$).

3. (**1 point**) Provide a simple proof (different from the one seen in class) that a function $f:\ \mathbb{R}^n \rightarrow \mathbb{R}$ that is convex in $\mathbb{R}^n$ and bounded is constant.

4. (**1 point**) Evaluate the support function $S_C(x)$ for the sets $C = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$ and $C = \mathcal{B}_2(0, 1)$ (ℓ_2-norm ball).

5. (**1 point**) Show that $f(x, y) = -\log(x^2 - ||y||_2^2)$ is convex in $\mathbf{dom} f = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n | x > ||y||_2\}$.

6. (**1 point**) Consider a quasi-linear function $f:\ \mathbb{R}^n \rightarrow \mathbb{R}$. What kind of convex sets are the superlevel and sublevel sets? (Hint: Use geometric intuition.)

7. (**1 point**) Calculate the dual function of $f(x) = x^p$ with $p > 1$ and $\mathbf{dom} f_0 = \mathbb{R}_+$ (Hint: Use the first-order condition $df_0(x)/dx = 0$ for optimization).

8. (**2 points**) Consider a convex set C and a real number $a \geq 0$.
a. Show that the set $S = \{x | \mathrm{dist}(x, C) \leq a\}$ is convex, where $\mathrm{dist}(x, C) = \inf_{y \in C} ||x - y||$.
b. Show that the set $T = \{x | B(x, a) \subseteq C\}$ is convex, where $B(x, a)$ represents a norm ball.