ECE 673-Random signal analysis I
Midterm - Oct. 25th 2006

Q1. Given the sample space $\mathcal{S}=\{(x,y): 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, are the two events $A = \{(x,y): 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \geq x\}$ and $B = \{(x,y): 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y < x\}$ independent?

Sol.: By drawing the two events it is apparent that they are mutually exclusive. Therefore, $P[A\,|B] = 0 \neq P[A] = 1/2$, which implies that A and B are not independent.

Q2. Is the average $E[X]$ the most probable value of a random variable X? Provide a thorough explanation (if needed, provide a counter-example).

Sol.: No, consider for instance a Bernoulli random variable with probability $p = 0.7$. In this case, the most probable value is 1 but the average is $E[X] = 0.7$.

Q3. In a country there are one million inhabitants. Among these, Joe knows 1000 persons. The same holds for Henry, whose acquaintances are selected independently from Joe. What is the average number of common acquaintances of Joe and Henry? (Hint: what is the probability that a person knows Joe? what is the probability that a person knows Henry? what is the probability that a person knows both Joe and Henry? What is the probability mass function of the number of common acquaintances?)

Sol.: The probability that a person knows Joe (or Henry) is $1000/(10^6) = 10^{-3}$.Therefore, the probability that a person knows both Joe and Henry is $p = 10^{-3} \cdot 10^{-3} = 10^{-6}$. Let us define $N = 10^6$, the number of common acquaintances is distributed according to a binomial distribution with parameters $bin(N,p)$. Therefore, the average is

$$Np = 10^{-6} \cdot 10^6 = 1.$$

Q4. Say that two random variables X and Y are uncorrelated. We consider a predictor of Y as

$$\hat{Y} = g(X)$$

where $g(\cdot)$ is a generic function. Can the following relationship hold true:

$$E[(Y - \hat{Y})^2] < var(Y)?$$

Provide a clear answer.

Sol.: Since X and Y are uncorrelated, we have $cov(X,Y) = 0$. In this case, we know that if $g(X)$ is constrained to be a linear function, $g(X) = aX + b$, then the mean square error $E[(Y - \hat{Y})^2] = var(Y)$. In other words, knowing X does not provide any additional useful information on Y. However, if X and Y are uncorrelated but not independent, there exists some non-linear function $g(X)$ so that the mean square error $E[(Y - \hat{Y})^2]$ is reduced with respect to $var(Y)$ (in particular, the optimal predictor is $E[Y\,|X]$).

P1. Two men play a game where they simultaneously select a random number in the set $\{0, 1, 2\}$ with equal probability.
(i) Assume that the choice is done independently by the two men. What is the joint probability mass function of the two numbers selected by the players?
(ii) What is the probability that they select the same number?
(iii) Now assume that the first player has the ability to predict, to a certain extent, the number that will be chosen by the second. In particular, the chooses 0 any time the second chooses 0. However, when the second chooses 1, he first chooses either 0 or 1 with equal probability. Finally, when the second chooses 2, the first is uncertain between 1 and 2 with equal probability. Find the joint probability mass function of the two numbers selected by the players (the second player still selects his number with equal probability in the set $\{0, 1, 2\}$). What is the probability that they select the same number?
(iv) Find the probability mass function of the random variable that measures the sum of the numbers selected by the two men (consider the scenario at point (i), where the numbers are independent).

Sol.: (i) Define X and Y as the numbers selected by the first and second players respectively. If the two variables are independent, then the joint probability mass function (PMF) is $p_{X,Y}[i, j] = p_X[i]p_Y[j]$. Since the marginals are simply equal to $p_X[k] = p_Y[k] = 1/3$ for $k = 0, 1, 2$, we have

$X\backslash Y$	0	1	2
0	1/9	1/9	1/9
1	1/9	1/9	1/9
2	1/9	1/9	1/9

(ii) The probability that the players select the same number reads

$$P[X = Y = 0] + P[X = Y = 1] + P[X = Y = 2] = 1/3.$$

(iii) From the problem statement, the conditional PMF of X given Y, $p_{X|Y}[x\,|y]$ reads

$X\backslash Y$	0	1	2
0	1	1/2	0
1	0	1/2	1/2
2	0	0	1/2

and the joint PMF is $p_{XY}[x, y] = p_{X|Y}[x\,|y]p_Y[y]$:

$X\backslash Y$	0	1	2
0	1/3	1/6	0
1	0	1/6	1/6
2	0	0	1/6

The probability that they select the same number is

$$P[X = Y = 0] + P[X = Y = 1] + P[X = Y = 2] = 2/3,$$

which is larger than that at point (ii), as expected.

(iv) The range of random variable Z is $\mathcal{S}_{\mathbf{Z}} = \{0, 1, 2, 3, 4\}$ and the PMF is

$$p_z[k] = \begin{cases} p_{XY}[0,0] = 1/9 & k = 0 \\ p_{XY}[1,0] + p_{XY}[0,1] = 2/9 & k = 1 \\ p_{XY}[1,1] + p_{XY}[2,0] + p_{XY}[0,2] = 1/3 & k = 2 \\ p_{XY}[2,1] + p_{XY}[1,2] = 2/9 & k = 3 \\ p_{XY}[2,2] = 1/9 & k = 4 \end{cases} .$$

P2. A random variable X is uniformly distributed between 2 and 3. We are interested in characterizing the transformed variable $Y = \ln X$.
(i) *Find and sketch* the probability density function of Y.
(ii) *Find and sketch* the cumulative distribution function of Y.
(iii) Write a small MATLAB code in order to estimate mean and variance of Y.

Sol.: (i) The range of Y is $\mathcal{S}_Y = (\ln 2, \ln 3)$ and the corresponding probability density function (PDF) of Y is obtained by using the known formula $p_Y(y) = p_X(g^{-1}(y)) \left| \partial g^{-1}(y)/\partial y \right|$, where we have used the fact that transformation $g(X) = \ln X$ is one-to-one. The inverse function reads $X = g^{-1}(Y) = e^y$. Recalling that $p_X(x) = 1$ for $2 < x < 3$ and $p_X(x) = 0$ elsewhere, we easily get

$$p_Y(y) = \begin{cases} e^y & \ln 2 < y < \ln 3 \\ 0 & \text{elsewhere} \end{cases} .$$

(ii) The corresponding cumulative distribution function (CDF) is obtained by using the definition $F_Y(y) = \int_{-\infty}^{y} p_Y(y) dy$:

$$F_Y(y) = \begin{cases} 0 & y \leq \ln 2 \\ e^y - 2 & \ln 2 < y < \ln 3 \\ 1 & y \geq \ln 3 \end{cases} .$$

(iii) Let us use the general inverse CDF method. Inverting the CDF we easily get

$$y = F_Y^{-1}(u) = \ln(u + 2) \text{ for } 0 \leq u \leq 1.$$

(Alternatively, we could have observed that $U + 2 \sim \mathcal{U}(2, 3)$).
Therefore, the corresponding MATLAB code reads

```
N=1000; %number of Monte Carlo simulations
EY=0; %initializing the estimate of the mean
EY2=0;%initializing the estimate of the average power
for n=1:N %for each Monte Carlo simulation
U=rand(1);
Y=log(U+2);
EY=EY+Y;
EY2=EY2+Y^2;
end
%evaluating the estimates
EYest=EY/N;
EY2est=EY2/N;
```

varYest=EY2est-EYest^2;

P3. The old saying goes that an apple a day keeps the doctor away, but now a group of scientists from University of Glasgow want to look into whether the same positive effect applies to apple cider. They set up an experiment where different people are given different amounts of apple cider per day and, for each person, the number of doctor's visits in a year are recorded. After having evaluated the histogram of the performed measures, the scientists found the following joint PMF for variables X, number of glasses of apple cider per day, and Y, number of doctor's visits per year:

$X\backslash Y$	0	1	2	3
0	0	0.15	0	0.1
1	0	0.15	0.1	0
2	0	0.15	0.1	0
3	0.1	0.15	0	0

(i) Are X and Y correlated? What is the correlation coefficient? Based on this result, what would you conclude about the effectiveness of apple cider?
(ii) Find the optimal linear predictor of Y given X

$$\hat{Y} = aX + b$$

and evaluate the corresponding mean square error.
(iii) Give a graphical interpretation of your result at the previous point.

Sol.: (i) The marginal PMFs read

$$p_X[k] = \begin{cases} 0.25 & k = 0 \\ 0.25 & k = 1 \\ 0.25 & k = 2 \\ 0.25 & k = 3 \end{cases}$$

and

$$p_Y[k] = \begin{cases} 0.1 & k = 0 \\ 0.6 & k = 1 \\ 0.2 & k = 2 \\ 0.1 & k = 3 \end{cases}.$$

The moments of interest read

$$\begin{aligned} E[X] &= 1.5 \\ var(X) &= E[X^2] - E[X]^2 = (1+4+9)/4 - 1.5^2 = 1.25 \\ E[Y] &= 0.6 + 2 \cdot 0.2 + 3 \cdot 0.1 = 1.3 \\ var(Y) &= E[X^2] - E[X]^2 = (1 \cdot 0.6 + 4 \cdot 0.2 + 9 \cdot 0.1) - 1.3^2 = 0.61 \\ E[XY] &= 0.15 + 2 \cdot 0.1 + 2 \cdot 0.15 + 4 \cdot 0.1 + 3 \cdot 0.15 = 1.5. \end{aligned}$$

The covariance reads

$$cov(X,Y) = E[XY] - E[X]E[Y] = 1.5 - 1.5 \cdot 1.3 = -0.45$$

and the correlation coefficient is

$$\rho = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{-0.45}{\sqrt{1.25 \cdot 0.61}} = -0.51.$$

Therefore, there is a negative, but not very strong correlation between X and Y.
(ii) The optimal linear predictor is

$$\begin{aligned} Y &= E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]) = 1.3 - \frac{0.45}{1.25}(X - 1.5) = \\ &= -0.36X + 1.84. \end{aligned}$$