ECE 755 - Digital communications
Midterm

Please provide clear and complete answers.

PART I: Questions -

Q.1. (1 point) A complex white Gaussian noise $W(t)$ with power per dimension $\sigma^2 = N_o/2$ is passed through a filter matched to the waveform $h(t) = e^{-t}u(t)$ ($u(t)$ is the unit-step function, $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ otherwise) and is sampled at rate $1/T$. Define as $N(t)$ the output of the filter. Find the autocorrelation $\rho_N(m) = E[N_k^* N_{k+m}]$ of the samples $N_k = N(kT)$ [Optional: find the power spectral density].

Sol.: We have

$$N_k =< W(t), h(t-kT) >= \int W(t)h(t-kT)dt,$$

which implies

$$\begin{aligned}
\rho_N(m) &= E[N_k^* N_{k+m}] = \int\int E[W^*(t)W(\tau)]h(t-kT)h(\tau-(k+m)T)dt = \\
&= \int\int N_o\delta(t-\tau)h(t-kT)h(\tau-(k+m)T)dt = \\
&= N_o \int h(t-kT)h(t-(k+m)T)dt = N_o\rho_h(m),
\end{aligned}$$

where

$$\begin{aligned}
\rho_h(m) &= \int h(t)h(t-mT)dt = \int h(t)h(t+mT)dt = \\
&\underset{m>0}{=} \int_0^\infty e^{-t}e^{-(t+mT)}dt \underset{m>0}{=} e^{-\tau}\int_0^\infty e^{-2t}dt \underset{m>0}{=} \frac{1}{2}e^{-mT}
\end{aligned}$$

so that by symmetry

$$\rho_h(m) = \frac{1}{2}e^{-|m|T} \rightarrow \rho_N(m) = \frac{N_o}{2}e^{-|m|T}.$$

Taking the z-transform and defining $\alpha = e^{-T}$, we obtain the power spectral density:

$$S_N(z) = \frac{1}{2}\frac{(1-\alpha^2)}{(1-\alpha z^{-1})(1-\alpha z)}.$$

Q.2. (1 point) The design of a radio communication system requires transmission of $R_b = 6Mb/s$ over a bandwidth of $W = 2MHz$ around a carrier frequency of $f_c = 2GHz$. Further specifications fix the constellation to be M-QAM and the roll-off factor $\alpha = 0.3$. Find the minimum required M and the corresponding symbol rate. Also, what is the minimum SNR E/N_o required for communications? Finally, what does it change if the system is implemented in baseband with a M-PAM constellation?

Sol.: The required spectral efficiency is

$$\nu = \frac{R_b}{W} = 3 \text{ bit/s/Hz}.$$

The required number of bits/ 2-dim is then

$$\rho = \nu(1+\alpha) = 3 \cdot 1.3 = 3.9 \text{ bit/2-dim},$$

the symbol rate:

$$\frac{1}{T} = \frac{W}{1+\alpha} = 1.54 \text{ Mbaud}$$

and the constellation size satisfies

$$|\mathcal{A}| \geq 2^{\rho} = 15 \rightarrow \text{16-QAM}.$$

The minimum E/N_o required for communications is

$$\frac{E}{N_o} = 2^{\rho} - 1 = 11.5\text{dB}.$$

If the system is in base-band, the constellation size becomes

$$|\mathcal{A}| \geq 2^{\rho/2} = 3.8 \rightarrow \text{4-PAM},$$

the symbol rate is

$$\frac{1}{T} = \frac{2W}{1+\alpha} = 3.08 \text{ Mbaud}$$

while the minimum energy remains the same.

Q.3. (1 point) Given two possible sequences $\mathbf{a}' = [-1\ 1\ 1]$ and $\mathbf{a}'' = [1\ -1\ 1]$ observed over an AWGN channel with noise power per dimension $\sigma^2 = N_o/2 = 0.1$, evaluate the likelihoods for $\mathbf{a}'$ and $\mathbf{a}''$ given the received signal samples $\mathbf{y} = [1.3\ -0.6\ 0.8]$ (you can neglect inessential multiplicative constants). Also, evaluate the ratio of the two likelihoods. Then assume that a priori probabilities for the sequences are $p(\mathbf{a}') = 0.8$ and $p(\mathbf{a}'') = 0.2$, and find the a posteriori probabilities of $\mathbf{a}'$ and $\mathbf{a}''$ given the same $\mathbf{y}$ (again, you can neglect inessential multiplicative constants), and the their ratio. What is the output of the MLSD and of the MAPSD?

Sol.: The likelihoods are as follows:

$$\begin{aligned}
f(\mathbf{y}|\mathbf{a}') &= f(y_0|a_0 = -1) f(y_1|a_1 = 1) f(y_2|a_2 = 1) = \\
&= \frac{1}{(2\pi \cdot 0.1)^{3/2}} \exp\left(-\frac{(1.3+1)^2 + (-0.6-1)^2 + (0.8-1)^2}{2 \cdot 0.1}\right) = 1.5 \cdot 10^{-17} \\
f(\mathbf{y}|\mathbf{a}'') &= \frac{1}{(2\pi \cdot 0.1)^{3/2}} \exp\left(-\frac{(1.3-1)^2 + (-0.6+1)^2 + (0.8-1)^2}{2 \cdot 0.1}\right) = 0.47
\end{aligned}$$

and the log-likelihood ratio is:

$$\log \frac{f(\mathbf{y}|\mathbf{a}')}{f(\mathbf{y}|\mathbf{a}'')} = \log \frac{1.5 \cdot 10^{-17}}{0.47} = -38.$$

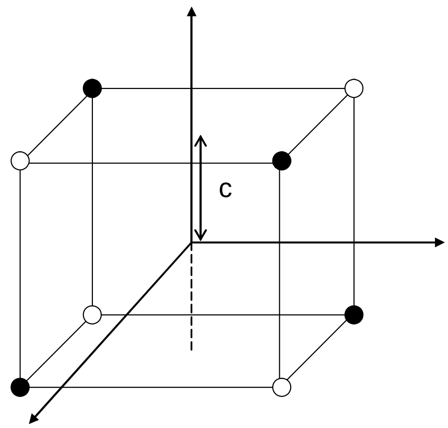

Figure 1:

A posteriori probabilities:

$$p(\mathbf{a}'|\mathbf{y}) = \frac{p(\mathbf{a}') \cdot f(\mathbf{y}|\mathbf{a}')}{f(\mathbf{y})} \propto p(\mathbf{a}') \cdot f(\mathbf{y}|\mathbf{a}') = 0.8 \cdot 1.5 \cdot 10^{-17} = 1.2 \cdot 10^{-17}$$

$$p(\mathbf{a}''|\mathbf{y}) = \frac{p(\mathbf{a}'') \cdot f(\mathbf{y}|\mathbf{a}'')}{f(\mathbf{y})} \propto p(\mathbf{a}'') \cdot f(\mathbf{y}|\mathbf{a}'') = 0.2 \cdot 0.47 = 0.094,$$

with log-likelihood ratio:

$$\log \frac{p(\mathbf{a}'|\mathbf{y})}{p(\mathbf{a}''|\mathbf{y})} = \log \frac{1.2 \cdot 10^{-17}}{0.094} = -36.6.$$

Both detectors select $\mathbf{a}''$.

Q.4. (1 point) Consider a spread spectrum (SS) system with BPSK modulation (recall that a SS signal is given by $\sum_n a_n g(t - nT)$ with waveform $g(t)$ having bandwidth-time product $WT = N > 1$). Find the spectral efficiency (bit/ 2-dim) and the SNR gap to capacity (SNR_{norm}) as a function of the spreading factor N and the SNR E/N_o. What happens when the spreading factor $N \to \infty$? Explain the result.
Sol.: Spectral efficiency

$$\rho = \frac{1}{N} \text{ [bit/ 2-dim]},$$

SNR gap to capacity:

$$SNR_{norm} = \frac{E/N_o}{2^\rho - 1} = \frac{E/N_o}{2^{\frac{1}{N}} - 1} \underset{N\to\infty}{\to} \infty.$$

This result shows that SS becomes increasingly inefficient in the use of the bandwidth as the spreading factor N increases.

PART II: Problems -

P.1. (2 points) Consider codes defined on the 3-dimensional hyper-cube of fig. 1 (the origin of the axes is at the center of the cube and we denote the length of each side as $2c$). Assume that the signal is received on a AWGN channel with power per dimension $\sigma^2 = N_o/2$.
(a) Study at first the code consisting of all the vertices (black and white). What is the spectral efficiency (number of bits per two dimensions)? Also, find the exact probability of symbol error and the union bound approximation as a function of the SNR E/N_o (recall that E is the energy per 2-dim).
(b) Now, consider the smaller code given by only white vertices. What is now the spectral efficiency? Find the union bound approximation of the symbol error probability. Is the exact probability of error easy to calculate in this case?
(c) Suppose that the SNR is $E/N_o = 6$dB and calculate the probability of error for the two codes (you can use the union bound approximations). Repeat with $E_b/N_o = 6$dB and for $SNR_{norm} = 6dB$. Draw some conclusions.

Sol.: (a) Spectral efficiency

$$\rho = \frac{2 \cdot \log_2 8}{3} = \frac{6}{3} = 2.$$

Probability of symbol error:

$$\begin{aligned}\Pr[\text{symbol error}] &= 1 - \left(1 - Q\left(\frac{2c}{\sqrt{2N_o}}\right)\right)^3 = \\ &= 1 - \left(1 - Q\left(\sqrt{\frac{E}{N_o}}\right)\right)^3,\end{aligned}$$

where we have used the fact that $E = 2c^2$. The union bound approximation is

$$\Pr[\text{symbol error}] \simeq 3Q\left(\sqrt{\frac{E}{N_o}}\right).$$

(b) Spectral efficiency:

$$\rho = \frac{2 \cdot \log_2 4}{3} = \frac{4}{3}.$$

Union bound approximation of the probability of symbol error:

$$\Pr[\text{symbol error}] \simeq 3Q\left(\frac{2\sqrt{2}c}{\sqrt{2N_o}}\right) = 3Q\left(\sqrt{\frac{2E}{N_o}}\right).$$

The exact probability of error is not easy to obtain given the shape of the decision regions.
(c) Fixing $E/N_o = 6dB$, we have

$$\begin{aligned}\Pr[\text{symbol error}]_{full} &\simeq 3Q\left(\sqrt{4}\right) = 0.07 \\ \Pr[\text{symbol error}]_{small} &\simeq 3Q\left(\sqrt{2 \cdot 4}\right) = 0.007\end{aligned}$$

The small constellation as a coding gain of 3dB at the cost of sacrificing 2/3 of the spectral efficiency of the full constellation.

With $E_b/N_o = 6dB$

$$\begin{aligned} \Pr[\text{symbol error}]_{full} &\simeq 3Q\left(\sqrt{\frac{E_b\rho}{N_o}}\right) = 3Q\left(\sqrt{\frac{2E_b}{N_o}}\right) = 0.007 \\ \Pr[\text{symbol error}]_{small} &\simeq 3Q\left(\sqrt{\frac{2E_b\rho}{N_o}}\right) = 3Q\left(\sqrt{\frac{8E_b}{3N_o}}\right) = 0.0016. \end{aligned}$$

With $SNR_{norm} = 6dB$

$$\begin{aligned} \Pr[\text{symbol error}]_{full} &\simeq 3Q\left(\sqrt{SNR_{norm}\cdot(2^{\rho}-1)}\right) = 3Q\left(\sqrt{3SNR_{norm}}\right) = 8\cdot 10^{-4} \\ \Pr[\text{symbol error}]_{small} &\simeq 3Q\left(\sqrt{2SNR_{norm}\cdot(2^{\rho}-1)}\right) = 3Q\left(\sqrt{2SNR_{norm}\cdot(2^{4/3}-1)}\right) = \\ &= 7.5\cdot 10^{-4} \end{aligned}$$

From these last two results, we see that, when normalizing the required energy by the rate or the minimum required energy (from Shannon' capacity), the two constellations have similar performance.

P.2. (2 points) A given ISI channel has folded spectrum $S_h(e^{j2\pi fT}) = 2\cdot(1-cz^{-1})(1-c^*z)$ with $|c|<1$. $L=3$ symbols are transmitted from a binary constellation $\mathcal{A}=\{0,1\}$.
(a) Draw the Whitened Matched filter front-end. Is the whitening filter realizable?
(b) Assume that $c=0$ and that the received signal is $\mathbf{y}=[0.4\ 0.8\ 0.9]$. Derive the optimal MLSD detector and find its output corresponding to input $\mathbf{y}$.
(c) Assume now $c=0.3$ and the received signal $\mathbf{y}=[0.4\ 0.8\ 0.9\ 1.3]$. Solve the problem at the previous point by using the Viterbi algorithm. Sketch carefully metrics and survivors at the different steps.
(d) Suppose that the all-zero sequence is transmitted and write the approximate probability of sequence error as a function of the noise power per dimension σ^2.

Sol.: (a) See textbook or notes for the block diagram. The whitening filter is

$$\frac{1}{\gamma^2 M^*(1/z^*)} = \frac{1}{2(1-c^*z)}.$$

which is IIR anti-causal (and stable). It is thus not realizable (unless $c=0$).
(b) If $c=0$, no ISI is present, and MLSD can be performed symbol by symbol, so that the optimal detector is a simple slicer with threshold 0.5. This leads to the MLSD estimate $\hat{\mathbf{a}}=[0\ 1\ 1]$.
(c) The equivalent channel after the whitener is $M(z)=1-0.3z^{-1}$, which leads to the trellis in fig. 2, where the branch metrics are also shown. The algorithm then proceeds as usual (see textbook or notes). The output is $\hat{\mathbf{a}}=[0\ 1\ 1]$.
(d) From the union bound approximation

$$\Pr[\text{sequence error}] \simeq 3Q\left(\frac{d_{\min}}{2\sigma}\right) = 3Q\left(\frac{\sqrt{1^2+0.3^2}}{2\sigma}\right) = 3Q\left(\frac{1.04}{2\sigma}\right)$$

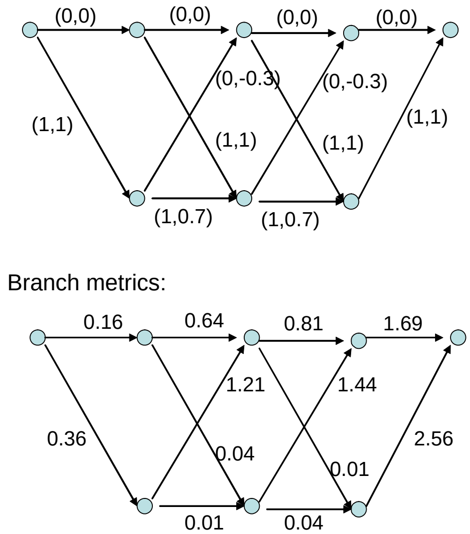

Figure 2:

P.3. (2 points) The a priori probabilities of two symbols a_0, $a_1 \in \mathcal{A} = \{0, 1\}$ satisfy $p_{A_0}(0) = 3/4$ and $p_{A_1}(0) = 1/2$. The symbols are transmitted over an ISI channel that is described by the transfer function $M(z) = 1 - 0.5z^{-1}$. We would like to obtain soft-information about the transmitted symbols given that the received signal is $\mathbf{z}$ =[1.1 0.6 −0.3] and knowing that the noise power is $\sigma^2 = N_o/2 = 0.5$ (we also have $a_k = 0$ for $k < 0$ and $k > 2$).
(a) Draw the trellis with appropriate branch metrics $\gamma_k(p, q)$ for the BCJR algorithm.
(b) Perform the BCJR algorithm and find the a posteriori probabilities of the transmitted symbols ($p_{A_k}(a|\mathbf{z})$).
(c) What are the log-likelihood ratios for the three symbols? Which symbol can be decided on with the highest confidence?

Sol.: (a) The branch metrics $\gamma_k(p, q)$ can be written as $\gamma_k(p, q) = p_{A_k}(a) f(z_k|\Psi_k = p, \Psi_{k+1} = p) = p_{A_k}(a) \exp(-(z_k - s_k^{(p,q)})/(2\sigma^2))$ where we have used standard notation and have dropped the constant $1/\sqrt{2\pi\sigma^2}$ since it is immaterial for the algorithm. The received signals $s_k^{(p,q)}$ corresponding to state transition (p, q) can be easily found, and so can the branch metrics $\gamma_k(p, q)$, as in Fig. 3.
(b) Forward and backward recursions are shown in the Fig. 4, along with the unnormalized and normalized values of $\sigma_k(p, q)$. The corresponding values of the a posteriori probabilities of the transmitted symbols are:

$$p_{A_o}(0|\mathbf{z}) = 0.52,\ p_{A_o}(1|\mathbf{z}) = 0.48$$
$$p_{A_1}(0|\mathbf{z}) = 0.33,\ p_{A_1}(1|\mathbf{z}) = 0.67$$

so that the log-likelihoods are:

$$\lambda_0 = \log \frac{p_{A_o}(1|\mathbf{z})}{p_{A_o}(0|\mathbf{z})} = -0.08$$
$$\lambda_1 = 0.7$$

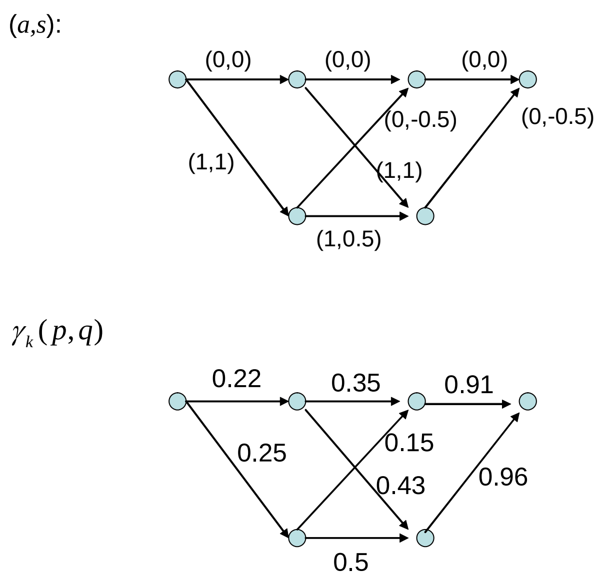

Figure 3:

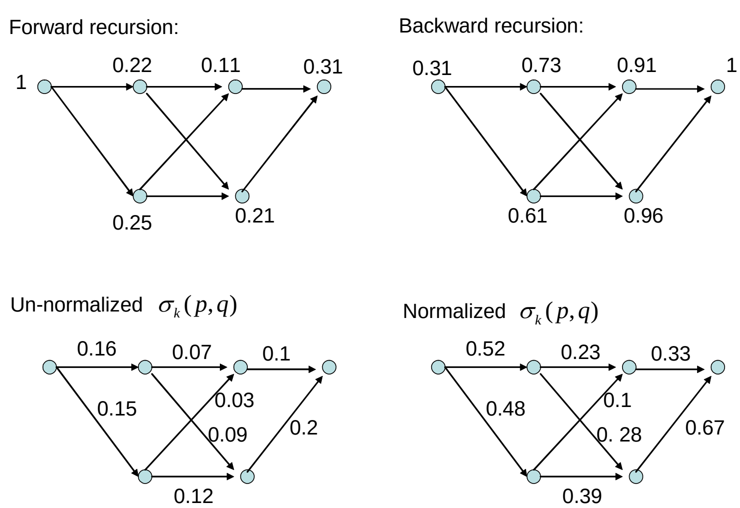

Figure 4: