Random signl analysis I (ECE673)
Solution assignment 6

1. Two independent discrete random variables X_1 and X_2 are known to have the following marginal PMF:

$$p_{X_i}(k) = \begin{cases} 1/2 & k=1 \\ 1/2 & k=2 \end{cases} \qquad i=1,2.$$

(i) Evaluate the joint PMF of X_1 and X_2: $p_{X_1,X_2}[k_1,k_2]$.
(ii) Then, define the transformed random variables

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

and calculate the range of the random vector $\mathbf{Y} = [Y_1\ Y_2]^T$ ($S_{\mathbf{Y}}$) and the joint PMF $p_{\mathbf{Y}}[k_1,k_2]$.
(iii) Moreover, evaluate the marginal PMFs $p_{Y_1}[k]$ and $p_{Y_2}[k]$. Are Y_1 and Y_2 independent?
Solution: (i) Since X_1 and X_2 are independent, we have $p_{X_1,X_2}[k_1,k_2] = p_{X_1}[k_1]p_{X_2}[k_2]$:

$X_2 \backslash X_1$	1	2
1	1/4	1/4
2	1/4	1/4

(double check: the sum of all probabilities is $1/4 \cdot 4 = 1$).
(ii) Now, we define the vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

The range of $\mathbf{Y}$ is $[2,3,4] \times [-1,0,1]$ and the joint PMF reads

$Y_2 \backslash Y_1$	2	3	4
−1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

(iii) From the joint PMF above, the marginals are easily calculated:

$$\begin{aligned} p_{Y_1}[k] &= \begin{cases} 1/4 & k=2 \\ 1/2 & k=3 \\ 1/4 & k=4 \end{cases} \\ p_{Y_2}[k] &= \begin{cases} 1/4 & k=-1 \\ 1/2 & k=0 \\ 1/4 & k=1 \end{cases} \end{aligned}.$$

Since, e.g.,

$$p_{Y_1,Y_2}[3,-1] = 1/4 \neq p_{Y_1}[3]p_{Y_2}[-1] = 1/2 \cdot 1/4 = 1/8,$$

the random variables Y_1 and Y_2 are not independent.

2. Two discrete random variables X_1 and X_2 have the following joint PMF $p_{\mathbf{X}}[x_1, x_2]$ (where $\mathbf{X} = [X_1\ X_2]^T$)

$X_2 \backslash X_1$	0	1	2
0	1/8	0	0
1	0	1/8	1/4
2	0	1/4	1/4

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(i) Evaluate the marginal PMFs $p_{X_1}[x_1]$ and $p_{X_2}[x_2]$.
(ii) Evaluate the covariance matrix $\mathbf{C_X}$ (i.e., the variances $var(X_1)$, $var(X_2)$ and the covariance $cov(X_1, X_2)$). Moreover, evaluate the correlation coefficient

$$\rho_{X_1X_2} = \frac{cov(X_1, X_2)}{\sqrt{var(X_1)var(X_2)}}.$$

Do you expect linear prediction (e.g., of X_1 given the measurement of X_2) to be effective?
(iii) Are X_1 and X_2 uncorrelated? Are X_1 and X_2 independent?
(iv) According to the results at the previous points, do you expect $var(X_1+X_2)$ to be larger than, smaller than or equal to $var(X_1) + var(X_2)$? Verify by computing $var(X_1 + X_2)$.
Solution:
(i) The marginal PMFs are esily evaluated as follows

$$\begin{aligned} p_{X_1}[k] &= \begin{cases} 1/8 & k = 0 \\ 3/8 & k = 1 \\ 1/2 & k = 2 \end{cases} \\ p_{X_2}[k] &= \begin{cases} 1/8 & k = 0 \\ 3/8 & k = 1 \\ 1/2 & k = 2 \end{cases} . \end{aligned}$$

(ii) We have

$$\begin{aligned} E[X_1] &= E[X_2] = 3/8 + 2 \cdot 1/2 = 11/8 = 1.375 \\ var(X_1) &= var(X_2) = E[X_1^2] - E[X_1]^2 = (1 \cdot 3/8 + 2^2 \cdot 1/2) - (11/8)^2 = 31/64 = 0.48 \\ cov(X_1, X_2) &= E[X_1X_2] - E[X_1]E[X_2] = \\ &= (1 \cdot 1 \cdot 1/8 + 1 \cdot 2 \cdot 1/4 + 2 \cdot 1 \cdot 1/4 + 2 \cdot 2 \cdot 1/4) - (11/8)^2 = 15/64 = 0.23 \end{aligned}$$

Therefore the covariance matrix is

$$\mathbf{C_X} = \begin{bmatrix} var(X_1) & cov(X_1, X_2) \\ cov(X_1, X_2) & var(X_2) \end{bmatrix} = \begin{bmatrix} 0.48 & 0.23 \\ 0.23 & 0.48 \end{bmatrix},$$

and the correlation coefficient reads

$$\rho_{X_1X_2} = \frac{cov(X_1, X_2)}{\sqrt{var(X_1)var(X_2)}} = \frac{15/64}{\sqrt{31/64 \cdot 31/64}} = \frac{15}{31} = 0.48.$$

Therefore linear prediction should help, to a certain extent, predicting one variable from the other.
(iii) X_1 and X_2 are not uncorrelated since $\rho_{X_1X_2} \neq 0$. Moreover, since they are not uncorrelated, they cannot be independent (if fact, independence implies uncorrelation).

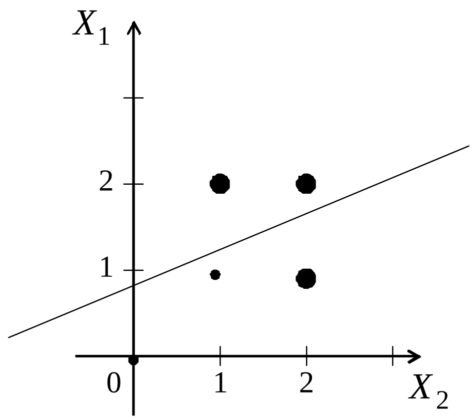

Figure 1:

(iv) Since $cov(X_1, X_2) > 0$, X_1 and X_2 vary in different directions (signs) around their means, $var(X_1 + X_2) > var(X_1) + var(X_2)$. In particular, we have

$$\begin{aligned} var(X_1 + X_2) &= \mathbf{1}^T \mathbf{C_X} \mathbf{1} = \\ &= var(X_1) + var(X_2) + 2cov(X_1, X_2) = \\ &= 31/64 + 31/64 + 2 \cdot 15/64 = 23/16 = 1.43. \end{aligned}$$

3. Consider the random variables described in the previous Problem. Say that we need to estimate X_1 from X_2 through a linear operation:

$$\hat{X}_1 = aX_2 + b,$$

where $\hat{X}_1$ denotes an estimate of X_1. Evaluate the estimator that minimizes the mean square error (i.e., calculate the optimal a and b) and provide a graphical interpretation in terms of a regression line. Moreover, calculate the estimation error and compare it with the variance of X_1. Explain your results.

Solution:

The optimal estimator is known to be

$$\begin{aligned} \hat{X}_1 &= \frac{cov(X_1, X_2)}{var(X_2)}(X_2 - E[X_2]) + E[X_1] = \\ &= \frac{15/64}{31/64}(X_2 - 11/8) + 11/8 = 15/31 X_2 + 22/31 = \\ &= 0.48X_2 + 0.71 \end{aligned}$$

The regression line is shown in the figure.

The corresponding estimation error (mean square error) reads

$$\begin{aligned} mse &= E[(\hat{X}_1 - X_1)^2] = \\ &= var(X_1) - \frac{cov(X_1, X_2)^2}{var(X_2)} = 31/64 - \frac{(15/64)^2}{31/64} = 23/62 = \\ &= 0.37 < var(X_1) = 0.48, \end{aligned}$$

where the last inequality is expected since the correlation coefficient $\rho_{X_1X_2}$ is not zero.