Random signl analysis I (ECE673)
Assignment 6

The due date for this assignment is Wednesday Oct. 18th.
Please provide detailed answers.
1. Two independent discrete random variables X_1 and X_2 are known to have the following marginal PMF:

$$p_{X_i}(k) = \begin{cases} 1/2 & k = 1 \\ 1/2 & k = 2 \end{cases} \qquad i = 1, 2.$$

(i) Evaluate the joint PMF of X_1 and X_2: $p_{X_1,X_2}[k_1, k_2]$.
(ii) Then, define the transformed random variables

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

and calculate the range of the random vector $\mathbf{Y} = [Y_1\ Y_2]^T$ $(S_{\mathbf{Y}})$ and the joint PMF $p_{\mathbf{Y}}[k_1, k_2]$.
(iii) Moreover, evaluate the marginal PMFs $p_{Y_1}[k]$ and $p_{Y_2}[k]$. Are Y_1 and Y_2 independent?

2. Two discrete random variables X_1 and X_2 have the following joint PMF $p_{\mathbf{X}}[x_1, x_2]$ (where $\mathbf{X} = [X_1\ X_2]^T$)

$X_2 \backslash X_1$	0	1	2
0	1/8	0	0
1	0	1/8	1/4
2	0	1/4	1/4

(i) Evaluate the marginal PMFs $p_{X_1}[x_1]$ and $p_{X_2}[x_2]$.
(ii) Evaluate the covariance matrix

$$\mathbf{C}_{\mathbf{X}} = \begin{bmatrix} var(X_1) & cov(X_1, X_2) \\ cov(X_1, X_2) & var(X_2) \end{bmatrix}$$

Moreover, evaluate the correlation coefficient

$$\rho_{X_1 X_2} = \frac{cov(X_1, X_2)}{\sqrt{var(X_1) var(X_2)}}.$$

Do you expect linear prediction (e.g., of X_1 given the measurement of X_2) to be effective?
(iii) Are X_1 and X_2 uncorrelated? Are X_1 and X_2 independent?

3. Consider the random variables described in the previous Problem. Say that we need to estimate X_1 from X_2 through a linear operation:

$$\hat{X}_1 = aX_2 + b,$$

where $\hat{X}_1$ denotes an estimate of X_1. Evaluate the estimator that minimizes the mean square error (i.e., calculate the optimal a and b) and provide a graphical interpretation in terms of a regression line. Moreover, calculate the estimation error and compare it with the variance of X_1. Explain your results.