Random signal analysis I (ECE673)
Assignment 5

The due date for this assignment is Wednesday Oct. 11

Please provide detailed answers.
1. Calculate the PDF of $Y = \exp(X)$ where $X \sim \mathcal{U}(0,1)$. Moreover, evaluate the average and variance of Y.
Solution: The range of Y is

$$\mathcal{S}_Y = \{y \colon 1 \le y \le e\}.$$

Moreover, the tranformation $g(x) = \exp(x)$ is clearly one-to-one (monotonically increasing). Therefore we can write

$$\begin{aligned} p_Y(y) &= p_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| = p_X(\log y) \left| \frac{\partial \log y}{\partial y} \right| = \\ &= p_X(\log y) \frac{1}{y}. \end{aligned}$$

But the PDF of X reads

$$p_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases},$$

thus the PDF of Y is

$$p_Y(y) = \begin{cases} 1/y & 1 \le y \le e \\ 0 & \text{elsewhere} \end{cases},$$

where $e = \exp(1)$, which is approximately 2.72. We have to check that $p_Y(y)$ is a PDF:

$$\int_{-\infty}^{\infty} p_Y(y) dy = \int_1^e \frac{1}{y} dy = [\log y]_1^e = 1.$$

The average of Y can be obtained according to the definition

$$E[Y] = \int_{-\infty}^{\infty} y p_Y(y) dy = \int_1^e dy = e - 1 = 1.72$$

or equivalently

$$E[Y] = \int_{-\infty}^{\infty} \exp(x) p_X(x) dx = \int_0^1 \exp(x) dx = e - 1 = 1.72.$$

The variance is $var(Y) = E[Y^2] - E[Y]^2$, where

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 p_Y(y) dy = \int_1^e y dy = \frac{1}{2} y^2 |_1^e = \frac{1}{2}(e^2 - 1) = 3.2$$

so that

$$var(Y) = E[Y^2] - E[Y]^2 = 3.2 - (1.72)^2 = 0.24.$$

2. Following the previous problem, estimate (i) the PDF of Y (i.e.,evaluate the histogram); (ii) the average $E[Y]$; (iii) the variance $var(Y)$ using MATLAB and compare your result with your answers at the previous point. Please include your MATLAB code and the obtained plot and outcomes.

Solution: A possible solution in MATLAB is as follows:

```
N=10000; %number of Monte Carlo iterations
dy=0.05;    %step size
bincenters=[-1:dy:exp(1)+1]; %bin centers used to calculate the histograms
bins=length(bincenters); %number of bins
h=zeros(bins,1); %initializing the variable that counts the number of realizations falling in
each bin
Ey=0; %initializing the estimate of Y
Ey2=0; %initializing the estimate of Y^2
for i=1:N   %for each Monte Carlo iteration
x=rand(1); y=exp(x);
  for k=1:bins    %for each bin
    if (y>(bincenters(k)-dy/2))&(y<=bincenters(k)+dy/2)
     h(k)=h(k)+1;
   end %if
   end %for
  Ey=Ey+y;
  Ey2=Ey2+y^2;
end %for
pyest=h/(N*dy); %calculate the histogram
%plotting the results
stem(bincenters,pyest); xlabel('y'); ylabel('p_Y(y)');
hold on;
z=[1:0.001:exp(1)];
plot(z,1./z,'–');
Ey=Ey/N  %calculate the average
Ey2=Ey2/N; %calculate the average of the power of Y
vary=Ey2-Ey^2
```

My outcome is shown in the figure.
and my numerical values are

Ey =
1.7252
vary =
0.2442

3. The signal-to-noise ratio (SNR) of a given measurement defines its accuracy. If X is a random variable modelling the measurement, the SNR is defined as $E[X]^2/var(X)$ and is seen to increase as the mean (which represents the true value to be measured) increases and/or the variance (which represents the power of the measurement error $X - E[X]$) decreases. (i)

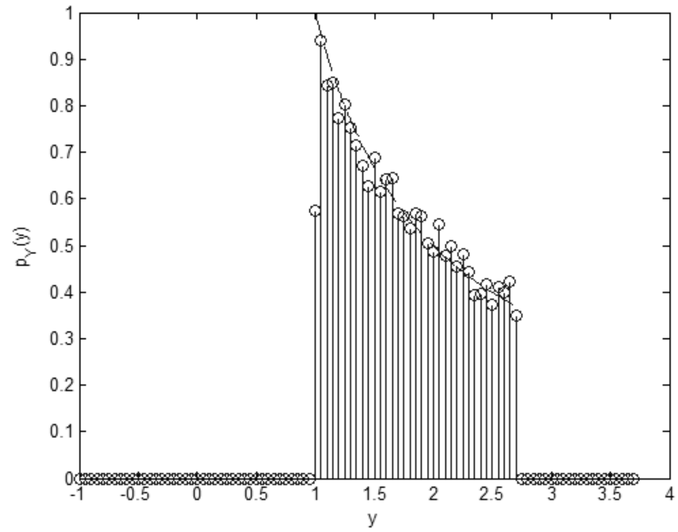

Figure 1:

Determine the SNR if the measurement is $X = A + U$ where A is the true value (a constant) to be measured and U model the measurement error with $U \sim \mathcal{U}(-1/2, 1/2)$. (ii) In order to have a SNR of 1000 what should A be? (iii) Say now that the measurement X is modelled by an exponential random variable (the true value to be measured is the mean $1/\lambda$). Explain why the SNR does not increase as the mean increases.

Solution: (i) The SNR reads

$$\begin{aligned} SNR &= \frac{E[X]^2}{var(X)} = \frac{(A + E[U])^2}{var(A+U)} = \frac{A^2}{var(U)} = \\ &= \frac{A^2}{1/12} = 12A^2. \end{aligned}$$

(ii) Let us impose the required condition and solve for A

$$SNR = 12A^2 = 1000 \rightarrow A = 9.12.$$

(iii) If $X \sim \exp(\lambda)$, then

$$SNR = \frac{E[X]^2}{var(X)} = \frac{(1/\lambda)^2}{1/\lambda^2} = 1.$$

Therefore, since the variance is equal to the square of the mean, a measurement modelled as an exponential random variable presents a SNR always equal to 1.