Random signal analysis I (ECE673)
Solution assignment 4

1. If $Y = 2X + 1$, where X is a Poisson random variable with $\lambda = 5$, find the set of possible values for Y ($\mathcal{S}_Y$) and the expression of the probability mass function of Y ($p_Y[y_i]$). Moreover, evaluate the variance of Y.
Solution: Since the set of possible values for X is $\mathcal{S}_X = \{0, 1, 2, ...\}$, the corresponding set for Y is $\mathcal{S}_Y = \{1, 3, 5, 7, ...\}$. Moreover, since $y_i = g(x_i) = 2x_i + 1$ is a one-to-one mapping, we have $g^{-1}(y_i) = \frac{y_i - 1}{2}$ for all $y_i \in S_Y$ and therefore

$$\begin{aligned} p_Y[k] &= \begin{cases} p_X\left(\frac{k-1}{2}\right) & \text{for } k = 1, 3, 5, ... \\ 0 & \text{for } k \neq \mathcal{S}_Y \end{cases} = \\ &= \begin{cases} e^{-\lambda}\frac{\lambda^{(k-1)/2}}{((k-1)/2)!} & \text{for } k = 1, 3, 5, ... \\ 0 & \text{for } k \neq \mathcal{S}_Y \end{cases} \end{aligned}$$

The variance of Y reads

$$var(Y) = 2^2 \cdot var(X) = 4 \cdot 5 = 20.$$

2. (i) Evalute the probability mass function (PMF) $p_Y[k]$ of $Y = X^2$ where X is a binomial random variable $bin(2, 0.2)$.
(ii) Write a MATLAB code that allows you to compare your result at point (i) with the estimate of $\hat{p}_Y[k]$ obtained through Monte Carlo iterations. Increasing the number of realizations (Monte Carlo simulations) improves the estimate?
(iii) Evaluate (through analysis) the averages $E[X]$ and $E[Y]$ and the variances $var(X)$ and $var(Y)$.
(iv) Modify your MATLAB code at point (ii) in order to obtain the estimates $\widehat{E[X]}$, $\widehat{E[Y]}$, $\widehat{var(X)}$ and $\widehat{var(Y)}$ through Monte Carlo simulations. Compare with your analysis at point (iii).
Solution: The probability density function of the binomial is

$$\begin{aligned} p_X[k] &= \binom{2}{k}(0.2)^k(0.8)^{2-k} \text{ for } k = 0, 1, 2 \\ &= \begin{cases} 0.64 & k = 0 \\ 0.32 & k = 1 \\ 0.04 & k = 2 \end{cases} \end{aligned}$$

The discrete random variable Y has range $\mathcal{S}_Y = \{0, 1, 4\}$ and the PMF reads

$$p_Y[k] = \begin{cases} 0.64 & k = 0 \\ 0.32 & k = 1 \\ 0.04 & k = 4 \end{cases}$$

Moreover, the average values of X and Y are

$$\begin{aligned} E[X] &= Np = 2 \cdot 0.2 = 0.4 \\ E[Y] &= \sum_{y_i} y_i p_Y[y_i] = 0 \cdot 0.64 + 1 \cdot 0.32 + 4 \cdot 0.04 = 0.48. \end{aligned}$$

Notice that we could have calculated $E[Y]$ also using

$$E[Y] = \sum_{x_i} x_i^2 p_X[x_i] = 0 \cdot 0.64 + 1 \cdot 0.32 + 4 \cdot 0.25 = 0.48.$$

The variances read

$$\begin{aligned}
var(X) &= E[X^2] - E[X]^2 = \sum_{x_i} x_i^2 p_X[x_i] - (0.4)^2 = 0.48 - 0.16 = 0.32 \\
var(Y) &= E[Y^2] - E[Y]^2 = \sum_{y_i} y_i^2 p_X[y_i] - (0.48)^2 = \\
&= (0.64 \cdot 0 + 0.32 \cdot 1 + 0.04 \cdot 16) - (0.48)^2 = 0.73
\end{aligned}$$

A possible MATLAB code to estimate the PMF of Y and the two averages and variances is as follows:

```
N=10000; %number of Monte Carlo iterations
h=zeros(3,1); %contains the relative frequencies of values (0,1,4) for random variable Y
Exest=0; %contains the estimated average of X
Eyest=0; %contains the estimated average of Y
Ex2est=0; %contains the estimated average of X^2
Ey2est=0; %contains the estimated average of Y^2
for i=1:N %for each Monte Carlo iteration
 u=rand(1);
  if (u<=0.64)
    x=0;
  elseif (u>0.64)&(u<=0.96)
     x=1;
   elseif (u>0.75)
    x=2;
  end %if
 y=x^2;
  %updating the estimate of the PMF of Y
  if (y==0) h(1)=h(1)+1;
   elseif (y==1) h(2)=h(2)+1;
   elseif (y==4) h(3)=h(3)+1;
  end %if
  %updating the average values
 Exest=Exest+x;
 Eyest=Eyest+y;
 %updating the average of X^2 and Y^2
 Ex2est=Ex2est+x^2;
 Ey2est=Ey2est+y^2;
end
 %dividing by the number of Monte Carlo simulations and showing the results for averages and PMF
Exest=Exest/N
Eyest=Eyest/N
```

pyest=h/N
%evaluating the estimate of the variances
Ex2est=Ex2est/N;
Ey2est=Ey2est/N;
varxest=Ex2est-Exest^2
varyest=Ey2est-Eyest^2
The MATLAB outcome is as follows:
Exest =
0.4057
Eyest =
0.4871
pyest =
0.6350
0.3243
0.0407
varxest =
0.3225
varyest =
0.7382
The estimates obtained through Monte Carlo iteration are pretty close to the real values.

3. (Problem 10.12) A constant or DC current source which outputs 1 Amp is connected to a resistor of resistance 1 Ohm. Due to measurement errors and sources of uncertainty such as the temperature, the current is better modelled as a random variable distributed according to $X \sim \mathcal{N}(0,1)$ (the average is the nominal value, the variance measures the squared measurement error). What is the probability that the voltage across the resistor is between -1 and 1 Volts? To answer this question, use MATLAB in order to evaluate the CDF of a standard Gaussian variable $\mathcal{N}(0,1)$ ($Q(x) = 1/2\,\text{erf}\,c(x/\sqrt{2})$).

Solution: Following the Ohm's law, the voltage across the resistor reads

$$V = R \times X\ Amp = X\ Volts,$$

and therefore

$$X \sim \mathcal{N}(0,1)\ [Volts].$$

The requested probability can be written as

$$P[-1 \leq X \leq 1].$$

that reads, recalling the definition of the CDF of a standard Gaussian variable $P[X \leq x] = \Phi(x) = 1 - Q(x)$:

$$P[-1 \leq X \leq 1] = \Phi(1) - \Phi(-1).$$

But since the Gaussian distribution is symmetric around the origin, we clearly have $\Phi(-1) = 1 - \Phi(1)$:

$$\begin{aligned} P[-1 &\leq X \leq 1] = 2\Phi(1) - 1 = \\ &= 1 - 2Q(1). \end{aligned}$$

Using $1/2\,\text{erf}\,c(1/\sqrt{2})$ =0.1587, we finally get

$$P[-1 \leq X \leq 1] = 1 - 2 \cdot 0.16 = 0.68.$$