Random signal analysis I (ECE673)
Assignment 1

1. A power meter measures the powers of two signals X_1 and X_2 and produces as an output the sum of the powers:

$$Y = X_1^2 + X_2^2.$$

Since there is uncertainty about the values of the input signals X_1 and X_2, we have to resort to a probabilistic model.
(i) Assuming that X_1 and X_2 are independent and Gaussian, write a MATLAB program in order to build a conjecture about the probability density function of the output Y ($p_Y(y)$). Towards this end, evaluate the histogram of Y using reasonable bin centers and bin size for $N = 1000$ Monte Carlo iterations. Recall that in order to generate independent and Gaussian random variables, you can use the commands: *x1=randn(1); x2=randn(1);* for each Monte Carlo iteration.
(ii) As we will learn during the course, it can be proved through analysis that the true probability density function is $p_Y(y) = 1/2 \cdot \exp(-y/2)$. Compare the results of your simulations with the true probability density function. What happens if we increase N?

Solution:

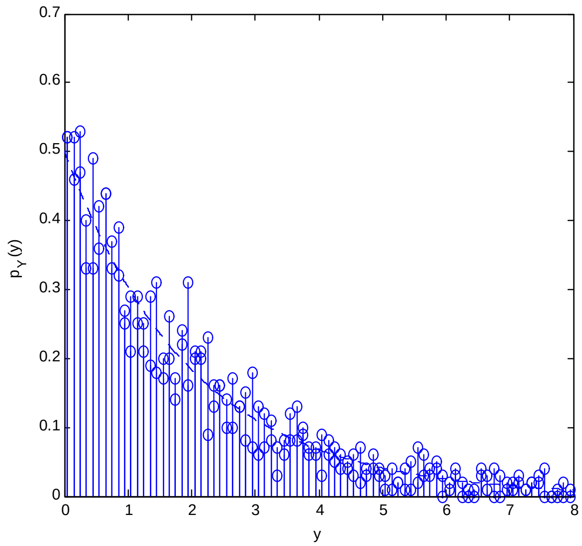
(i) Let us fix $N = 1000$ Monte Carlo iterations and a bin size $\Delta y = 0.1$. Moreover, since Y is positive, we can choose the bin centers $[\Delta y/2 : \Delta y : 8]$. The following MATLAB code estimates the probability density function of Y based on N measurements (realizations) of Y:

```
N=10000;
dy=0.1;
bincenters=[dy/2:dy:8];
bins=length(bincenters);
h=zeros(bins,1);
for i=1:N                                   %for each Monte Carlo iteration
 x1=randn(1);
 x2=randn(1);
 y=x1^2+x2^2;
   for k=1:bins                             %for each bin
      if (y>(bincenters(k)-dy/2))&(y<=(bincenters(k)+dy/2))
           h(k)=h(k)+1;
      end
   end
end
pyest=h/(N*dy);
stem(bincenters,pyest); xlabel('y'); ylabel('p_Y(y)');
```

(ii) In order to perform the comparison you can use the MATLAB code.

```
hold on; z=[0:0.01:8];
plot(z,1/2*exp(-z/2),'–');
```

The plot is shown in the figure below. Increasing the number of observations N would improve the accuracy of the estimate (try!).



2. (Problem 3.21) A die is tossed that yields an even number with twice the probability of yielding an odd number. What is the probability of obtaining an even number, an odd number, a number that is even or odd, a number that is even and odd?
Solution: The sample space is $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Moreover, in order to fully specify the probabilistic model, we need to assign probabilities to the simple events. From the problem statement, it is known that

$$\begin{aligned} P[\{2\}] &= P[\{4\}] = P[\{6\}] = 2p \\ P[\{1\}] &= P[\{3\}] = P[\{5\}] = p. \end{aligned}$$

But from the probability axioms, we have

$$1 = P[\mathcal{S}] = P[\cup_{i=1}^{6}\{i\}] = \sum_{i=1}^{6} P[\{i\}] = 3 \times 2p + 3p = 9p,$$

therefore it is

$$p = \frac{1}{9}.$$

Then, the probability of different events can be computed as the sum of the probabilities of simple events and from the basic properties of the probability function:

$$\begin{aligned} P[\{\text{even number}\}] &= P[\{2\}] + P[\{4\}] + P[\{6\}] = \frac{2}{3} \\ P[\{\text{odd number}\}] &= 1 - P[\{\text{even number}\}] = \frac{1}{3} \\ P[\{\text{odd number}\} \cup \{\text{even number}\}] &= \frac{2}{3} + \frac{1}{3} = 1 \\ P[\{\text{odd number}\} \cap \{\text{even number}\}] &= P[\emptyset] = 0. \end{aligned}$$

3. [EXTRA PROBLEM] (Problem 3.24) For a sample space $\mathcal{S} = \{0, 1, 2, ...\}$, the probability assignment

$$P[i] = \exp(-2)\frac{2^i}{i!}$$

is proposed. Is this a valid alignment?

Solution: It is necessary to verfiy that the sum over the probabilities of the simple events is one. We have

$$\sum_{i=1}^{+\infty} P[i] = \exp(-2)\sum_{i=1}^{+\infty}\frac{2^i}{i!} = \exp(-2)\exp(2) = 1,$$

since

$$\exp(x) = \sum_{i=1}^{+\infty}\frac{x^i}{i!}$$

is the Taylor series expansion of the exponential function about $x_0 = 0$.