ECE 788 - Optimization for wireless networks
Final

Please provide clear and complete answers.

PART I: Questions -

Q.1. Discuss an iterative algorithm that converges to the solution of the problem

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f_o(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \end{array},$$

where $f_o(\mathbf{x})$ is a strictly convex function in $\mathbb{R}^n$. Specify the algorithm by explicitly writing down the updating equations.

Sol.: A convenient choice is Newton's algorithm for equality-constrained problems, which is described by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + t^{(k)}\Delta\mathbf{x}^{(k)},$$

where the search direction $\Delta\mathbf{x}^{(k)}$ satisfies the linear system (along with the multipliers $\boldsymbol{\nu}$)

$$\begin{bmatrix} \nabla^2 f_o(\mathbf{x}^{(k)}) & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}^{(k)} \\ \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} -\nabla f_o(\mathbf{x}^{(k)}) \\ \mathbf{0} \end{bmatrix},$$

and $t^{(k)}$ are appropriately chosen step sizes (e.g., obtained via backtracking line search).

Q.2. You are given the following problem:

$$\begin{array}{ll} \underset{\mathbf{x},\mathbf{y}}{\text{minimize}} & \mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \begin{cases} \mathbf{C}\mathbf{x} \preceq \mathbf{d} \\ \mathbf{E}\mathbf{y} \preceq \mathbf{f} \\ \mathbf{G}\mathbf{x} + \mathbf{H}\mathbf{y} \preceq \mathbf{p} \end{cases} \end{array}$$

where $\mathbf{x}, \mathbf{y}$ are optimization variables in $\mathbb{R}^n$. In order to ease the solution, you are asked to distribute the optimization task over two subprocessors coordinated by a master processor. Propose both a primal and a dual decomposition solution to accomplish this task.

Sol.: Primal decomposition:

$$\underset{\mathbf{z}}{\text{minimize}} \quad \inf_{\mathbf{x}\in\mathbf{C}_x}(\mathbf{a}^T\mathbf{x}) + \inf_{\mathbf{y}\in\mathbf{C}_y}(\mathbf{b}^T\mathbf{y}),$$

with $\mathbf{C}_x = \{\mathbf{x}\text{: } \mathbf{C}\mathbf{x} \preceq \mathbf{d},\ \mathbf{G}\mathbf{x} \preceq \mathbf{z}\}$ and $\mathbf{C}_y = \{\mathbf{y}\text{: } \mathbf{E}\mathbf{y} \preceq \mathbf{f},\ \mathbf{H}\mathbf{y} \preceq \mathbf{p} - \mathbf{z}\}$.

Dual decomposition:

$$\underset{\boldsymbol{\lambda}}{\text{maximize}} \quad \inf_{\mathbf{x}\in\mathbf{C}'_x}(\mathbf{a}^T\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{G}\mathbf{x}) + \inf_{\mathbf{y}\in\mathbf{C}'_x}(\mathbf{b}^T\mathbf{y} + \boldsymbol{\lambda}^T\mathbf{H}\mathbf{y}) - \boldsymbol{\lambda}^T\mathbf{p},$$

with $\mathbf{C}'_x = \{\mathbf{x}\text{: } \mathbf{C}\mathbf{x} \preceq \mathbf{d}\}$ and $\mathbf{C}'_y = \{\mathbf{y}\text{: } \mathbf{E}\mathbf{y} \preceq \mathbf{f}\}$.

Q.3. Consider the strategic game defined by the following utility matrix:

$2,2$	$0,1$
$1,0$	$1,1$

.

Find Pareto-optimal points and Nash equilibria. Repeat for the game defined by the utility matrix:

$3,3$	$1,4$
$4,1$	$0,0$

.

Sol.: First game: Pareto-optimal point: $(2,2)$; Nash equilibria: $\{(2,2),(1,1)\}$.
Second game: Pareto-optimal points: $\{(3,3),(1,4),(4,1)\}$; Nash equilibria: $\{(1,4),(4,1)\}$

Q.4. Consider the paper-rock-scissor game defined by the utility matrix

	P	R	S
P	$0,0$	$1,-1$	$-1,1$
R	$-1,1$	$0,0$	$1,-1$
S	$1,-1$	$-1,1$	$0,0$

Is there a Nash equilibrium in mixed strategies? If so, calculate it.

Sol.: For any finite- and discrete- strategy game there is at least a NE. It can be calculated via the indifference principle (denoting by p_{ij} the probability that player i chooses the option j)

$$\begin{aligned} E_p[U_1(x)|x_1 &= \mathrm{P}] = E_p[U_1(x)|x_1 = \mathrm{R}] = E_p[U_1(x)|x_1 = \mathrm{S}] \\ &\to p_{22} - (1 - p_{21} - p_{22}) = -p_{21} + (1 - p_{21} - p_{22}) = p_{21} - p_{22} \\ &\to p_{2j} = 1/3 \end{aligned}$$

and the same can be done for the utility of the second player obtaining $p_{1j} = 1/3$.

PART II: Problems -

P.1. For each of the following three problems: (a) solve the primal problem (i.e., give p^* and X_{opt}); (b) solve the dual problem (i.e., give d^* and the optimal multipliers); (c) are Slater's conditions satisfied?; (d) does strong duality hold?; (e) is there a solution to the KKT conditions?
1.

$$\begin{aligned} &\underset{x}{\text{minimize}}\ x \\ &\text{s.t.}\ x^2 \le 1 \end{aligned}$$

Sol.: Primal:

$$p^* = -1,\ X_{opt} = \{-1\}$$

Dual:

$$\begin{aligned} L(x,\lambda) &= x + \lambda(x^2-1) \\ \underset{x=-\frac{1}{2\lambda}}{\longrightarrow} g(\lambda) &= -\frac{1}{4\lambda} - \lambda \text{ with } \operatorname{dom} g = \{\lambda \neq 0\} \end{aligned}$$

The solution of the dual problem $\underset{\lambda>0}{\text{maximize}} -\frac{1}{4\lambda}-\lambda$ can be easily found by setting $\nabla g(\lambda) = 0$, which leads to $\lambda^* = 1/2$ and $d^* = -1$. Slater's conditions are satisfied (consider for instance $x = 0$ for which $x^2 < 1$), so that we know that strong duality holds and the dual problem is attained. KKT conditions: since the KKT conditions are necessary and sufficient for optimality, the only solution is $(x^*, \lambda^*) = (-1, 1/2)$.

2.

$$\begin{aligned} &\underset{x}{\text{minimize}}\ x \\ &\text{s.t. } x^2 \leq 0 \end{aligned}$$

Sol.: Primal:

$$p^* = 0, \ X_{opt} = \{0\} = \mathcal{C},$$

where $\mathcal{C}$ represents the feasible set.
Dual:

$$\begin{aligned} L(x,\lambda) &= x + \lambda x^2 \\ \underset{x=-\frac{1}{2\lambda}}{\longrightarrow} g(\lambda) &= -\frac{1}{4\lambda} \text{ with } \operatorname{dom} g = \{\lambda \neq 0\} \end{aligned}$$

The solution of the dual problem $\underset{\lambda>0}{\text{maximize}} -\frac{1}{4\lambda}$ is clearly $d^* = 0$ which is not attained. Slater's conditions are not satisfied ($\mathcal{C} = \{0\}$ for which $x^2 = 0$). Strong duality holds since $p^* = d^*$. KKT conditions: since Slater's conditions are not satisfied but the problem is convex, the KKT conditions are necessary and sufficient for (x, λ) to be primal-dual optimal with zero duality gap. Since here we know that there is no such (x, λ) (the dual problem is not attained), then the KKT have no solution.

3.

$$\begin{aligned} &\underset{x}{\text{minimize}}\ x \\ &\text{s.t. } |x| \leq 0 \end{aligned}$$

Sol.: Primal:

$$p^* = 0, \ X_{opt} = \{0\} = \mathcal{C},$$

where $\mathcal{C}$ represents the feasible set.
Dual:

$$\begin{aligned} L(x,\lambda) &= x + \lambda|x| \\ \underset{x=0}{\longrightarrow} g(\lambda) &= 0 \text{ with } \operatorname{dom} g = \{\lambda > 1\} \end{aligned}$$

We have $d^* = 0$ and the set of optimal values for λ is $\{\lambda > 1\}$. Slater's conditions are not satisfied ($\mathcal{C} = \{0\}$ for which $|x| = 0$). Strong duality holds since $p^* = d^*$. KKT conditions: the constraint is not differentiable so that we cannot write the KKT conditions.

P.2. In the multihop network in the figure below, each ith link has capacity C_i (in bits/sec) and it is operated for a fraction $0 \leq x_i \leq 1$ of the total time ($\sum_{i=1}^{M} x_i = 1$). The effective rate on the ith link is then x_iC_i. Moreover, the end-to-end rate from node 1 to node $M+1$ is easily shown to be $\min_{i=1,2,...,M}\{x_iC_i\}$, that is, it is limited by the worst link.

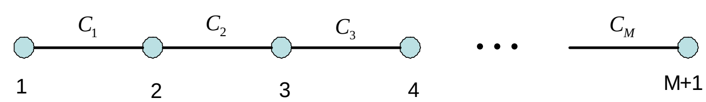

Figure 1:

(a) Write the problem of maximizing the end-to-end rate ($\min_{i=1,2,...,M}\{x_iC_i\}$) over the fraction of times $\mathbf{x}$. Is this problem convex? Does it satisfy Slater's conditions?

Sol.: The problem is

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \;\; & \min_{i=1,2,...,M}\{x_iC_i\} \\ \text{s.t.} \;\; & x_i \leq 1, \;\; -x_i \leq 0, \\ & \sum_{i=1}^{M} x_i = 1 \end{aligned}$$

which is easily shown to be convex since the constraints are affine and the objective is concave (the pointwise minimum of affine functions is concave). Moreover, since all the constraints are affine and the problem is feasible, it satisfies Slater's conditions.

(b) Write the problem as LP and the corresponding KKT conditions. Verify that the only solution of the KKT conditions is $x_j = \frac{1/C_j}{\sum_{i=1}^{M}\frac{1}{C_i}}$ for all $j = 1, ..., M$. (*Hint*: can an optimal solution have $x_i = 0$ for some i?) Finally, can we have optimal solutions for the problem at hand other than $x_j = \frac{1/C_j}{\sum_{i=1}^{M}\frac{1}{C_i}}$? Why?

Sol.: The equivalent LP problem is

$$\begin{aligned} \underset{\mathbf{x},t}{\text{minimize}} \;\; & -t \\ \text{s.t.} \;\; & \begin{cases} t - x_iC_i \leq 0 \\ \sum_{i=1}^{M} x_i = 1 \\ -x_i \leq 0 \end{cases} \end{aligned}$$

Noticing that the optimal solution must have $x_i > 0$ (otherwise the end-to-end rate would be zero), we can write the Lagrangian as

$$L(t, \mathbf{x}, \boldsymbol{\lambda}, \nu) = -t + \sum_{i=1}^{M} \lambda_i (t - x_i C_i) + \nu \left(1 - \sum_{i=1}^{M} x_i\right)$$

The KKT conditions are:

$$\begin{aligned}
1 - \sum_{i=1}^{M} \lambda_i &= 0 && \text{(1a)}\\
\lambda_i C_i + \nu &= 0 && \text{(1b)}\\
\sum_{i=1}^{M} x_i &= 1 && \text{(1c)}\\
t - x_i C_i &\leq 0 && \text{(1d)}\\
x_i &> 0 && \text{(1e)}\\
\lambda_i (x_i C_i - t) &= 0 && \text{(1f)}\\
\lambda_i &\geq 0 && \text{(1g)}
\end{aligned}$$

From (1b) and (1a) we have $\nu = \frac{1}{\sum_{i=1}^{M} \frac{1}{C_i}}$ and $\lambda_i = \frac{\nu}{C_i} > 0$. Since $\lambda_i > 0$, from (1f) and (1c), we obtain $t = \frac{1}{\sum_{i=1}^{M} \frac{1}{C_i}}$ and $x_i = \frac{1/C_i}{\sum_{j=1}^{M} \frac{1}{C_j}}$. Since the problem satisfies the Slater's conditions, KKT conditions are necessary and sufficient for optimality, so that no other optimal solutions can be found.

(c) Consider now an alternative scenario where C_i is not a constant but depends on x_i as

$$C_i(x_i) = \log\left(1 + \frac{P}{N x_i}\right), \tag{2}$$

where signal (P) and noise (N) powers are given and fixed. Equation (2) accounts for the fact that if a link is operated for less time it can employ more power (P/x_i) without violating the long-term power constraint of P. Is the problem of maximizing the capacity $\min_{i=1,2,\dots,M}\{x_i C_i(x_i)\}$ convex?

Sol.: Yes, since $x_i C_i(x_i)$ is concave (it can be proved by taking the second derivative or using the fact that it is the perspective function of $\log\left(1 + \frac{P}{N}\right)$, which is concave).

P.3. Two nodes transmit packets at rates x_1 and x_2 respectively on a given wired network. Node 1 has a greater need of bandwidth so that the utilities of the two nodes are $U_1(x_1) = 2x_1$ and $U_2(x_2) = x_2$, respectively. Moreover, the finite capacity of the links in the network poses the following constraints on $x_1 \geq 0$ and $x_2 \geq 0$:

$$\begin{aligned}
2x_1 + x_2 &\leq 1 && \text{(3a)}\\
x_1 + 2x_2 &\leq 1. && \text{(3b)}
\end{aligned}$$

(a) Plot the region of achievable utility values θ, and identify the Pareto optimal points.

Sol.: The region θ of achievable utility values is given by the equations ($U_1 \geq 0$ and $U_2 \geq 0$):

$$\begin{aligned} U_1 + U_2 &\leq 1 \\ \frac{U_1}{2} + 2U_2 &\leq 1, \end{aligned}$$

from the problem constraints and the definitions of the utility functions. It follows that region θ is a polytope with vertices $(0,0), (1,0), (2/3, 1/3), (0, 1/2)$. The Pareto optimal points are easily shown to be given by the equations:

$$\begin{aligned} U_2 &= \frac{1}{2} - \frac{U_1}{4} \text{ for } 0 \leq U_1 \leq 2/3 \\ U_2 &= 1 - U_1 \text{ for } 2/3 \leq U_1 \leq 1. \end{aligned}$$

(b) Consider the scalarization method for finding the Pareto-optimal points. Are all the Pareto-optimal points solutions of a scalar problem? If so, give the corresponding values of the weights $\boldsymbol{\alpha}$ that provide the Pareto-optimal points.

Sol.: Since the problem is convex (utilities and constraints are linear), all the Pareto optimal points are solutions of the scalar problem

$$\begin{aligned} &\underset{\mathbf{x}}{\text{maximize}}\ \alpha_1 U_1(\mathbf{x}) + \alpha_2 U_2(\mathbf{x}) \\ &\text{s.t.} \quad \begin{array}{l} x_1 \geq 0,\ x_2 \geq 0 \\ 2x_1 + x_2 \leq 1 \\ x_1 + 2x_2 \leq 1. \end{array} \end{aligned}$$

for some $\boldsymbol{\alpha} \succeq 0$. In particular, it is clear that by choosing $\boldsymbol{\alpha} = [1\ 1]^T$ and $\boldsymbol{\alpha} = [1/2\ 2]^T$ (or scalar multiples of these vectors), we obtain all the Pareto-optimal points.

(c) Formulate this scenario as a strategic game, where the strategy set is defined by all the $x_1 \geq 0$ and $x_2 \geq 0$ that satisfy (3). Is there at least a Nash equilibrium? If so, find the Nash equilibria.

Sol.: The game at hand is $< \{1, 2\}, \mathcal{C}, \{U_i(\mathbf{x})\}_{i=1,2} >$, where the set of strategy $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}_+^2 : 2x_1 + x_2 \leq 1,\ x_1 + 2x_2 \leq 1\}$. Since the set of strategies is compact and the utility $U_i(\mathbf{x})$ is quasiconcave in x_i and continuous in $\mathbf{x}$, we know that the game has at least one Nash equilibrium. From the definition of Nash equilibrium, it is easy to see that all the Pareto optimal points of the problem at hand are also Nash equilibria.