ECE 788 - Optimization for wireless networks
Final

Please provide clear and complete answers.

1. (4 points) Consider the optimization problems P1, P2, P3 and P4 below, where $f_o(x)$ is the cost function and $f_1(x)$ defines the inequality constraint (i.e., the problem is "minimize $f_o(x)$ s.t. $f_1(x) \leq 0$"):
P1. $f_o(x) = x$, $f_1(x) = |x|$ with domain $\mathcal{D} = \mathbb{R}$;
P2. $f_o(x) = x^3$, $f_1(x) = -x + 1$ with domain $\mathcal{D} = \mathbb{R}$;
P3. $f_o(x) = x^3$, $f_1(x) = -x + 1$ with domain $\mathcal{D} = \mathbb{R}^+$;
P4. $f_o(x) = x$, $f_1(x) = \begin{cases} -x-2 & \text{for } x \leq -1 \\ x & \text{for } -1 \leq x \leq 1 \\ -x+2 & \text{for } x \geq 1 \end{cases}$ with domain $\mathcal{D} = \mathbb{R}$.
a. For all the problems above, state whether the problem is convex and, if so, whether Slater's condition holds.
b. For all the problems above, derive and plot the perturbation function $p(u)$, and identify p^*, d^* along with x^* and λ^* if they exist (please provide all the necessary details).

Sol.:
P1. a. The problem is convex, and Slater's condition does not hold.
b. The perturbation function is $p(u) = -u$ with $dom p = \{u : u \geq 0\}$. We thus have $p^* = d^* = 0$ with $x^* = 0$ and $\lambda^* = 1$ (more precisely, any $\lambda^* \geq 1$ is dual optimal).

P2. a. The problem is not convex.
b. The perturbation function is $p(u) = (1-u)^3$ with $dom p = \mathbb{R}$. We thus have $p^* = 1$, while $d^* = -\infty$, with $x^* = 1$ (the value of d^* can of course also double checked by solving the dual problem).

P3. a. The problem is convex and Slater's condition is satisfied.
b. The perturbation function is $p(u) = (1-u)^3$ for $u \leq 1$ and $p(u) = 0$ for $u > 1$. We thus have $p^* = 1$, while $d^* = 1$, with $x^* = 1$ and $\lambda^* = -dp(0)/du = 3$.

P4. a. The problem is not convex.
b. The perturbation function is $p(u) = -2 - u$ with $dom p = \{u : u \geq -1\}$. We thus have $p^* = d^* = -2$, with $x^* = -2$ and $\lambda^* = 1$.

2. (2 points) Consider the following problem

$$\begin{array}{c} \text{minimize } ||Ax - b||_2^2 \\ \text{s.t. } Gx = h \end{array},$$

with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ with $rank(A) = n$ and $G \in \mathbb{R}^{p \times n}$ with $rank(G) = p$.
a. Calculate the dual function.
b. Write the KKT conditions. Do you expect to be able to find a solution?
c. Find the optimal multiplier vector and the optimal solution as a function of the optimal multiplier.

Sol.: a. The Lagrangian function is

$$\begin{aligned}\mathcal{L}(x,\nu) &= ||Ax-b||_2^2 + \nu^T(Gx-h) \\ &= xA^TAx + (G^T\nu - 2A^Tb)^Tx - \nu^Th,\end{aligned}$$

and the dual function is thus obtained by minimizing the above strictly convex function, obtaining

$$g(\nu) = -\frac{1}{4}(G^T\nu - 2A^Tb)^T(A^TA)^{-1}(G^T\nu - 2A^Tb) - \nu^Th.$$

b. The KKT conditions are

$$\begin{aligned}2A^T(Ax^* - b) + G^T\nu^* &= 0 \\ Gx^* &= h.\end{aligned}$$

Since the problem is convex and satisfies Slater's conditions (it is feasible), an optimal point exists if and only if the KKT have a solution. From Weierstrass theorem, an optimal solution exists, and therefore the KKT must have a solution.
c. From the first KKT condition, we calculate

$$x^* = (A^TA)^{-1}(A^Tb - (1/2)G^T\nu^*),$$

which gives us from the second equation:

$$\nu^* = -2(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb).$$

3. (**2 points**) Consider a convex problem characterized by cost function $f_o(x)$, inequality constraints $f_i(x) \leq 0$, $i = 1, ..., m$ and equality constraints $h_i(x) = 0$, $i = 1, ..., p$. Prove that the perturbation function $p(u,v) = \inf f_o(x)$, where the infimum is taken under the constraints $f_i(x) \leq u_i$ and $h_i(x) = v_i$ ($u = (u_1, ..., u_m)$, $v = (v_1, ..., v_p)$), is a convex function. Recall that you have to prove also that the domain is a convex set.

Sol.: Consider the following function

$$g(x,u,v) = \begin{cases} \tilde{f}_o(x) & \begin{array}{c}\text{if } f_i(x) \leq u_i \\ \text{and } h_i(x) = v_i\end{array} \\ \infty & \text{otherwise}\end{cases},$$

where $\tilde{f}_o(x)$ is the extended value extension of $f_o(x)$. The function $g(x,u,v)$ is convex in x, u, v, as it can be shown from the definition of convex function. To see this, define $y = (x,u,v)$. Now consider any y_1, y_2 and a convex combination $y = \theta y_1 + (1-\theta)y_2$ $(0 \leq \theta \leq 1)$. We need to show that

$$g(y) \leq \theta g(y_1) + (1-\theta)g(y_2).$$

There are two cases. 1) If $g(y) = \infty$, then necessarily we must have $g(y_1) = \infty$ and/or $g(y_2) = \infty$. In fact, if $g(y_1)$ and $g(y_2)$ were finite, then, by the convexity of $\tilde{f}_o(x)$ and $f_i(x)$ and the fact that $h_i(x)$ is affine, $g(y)$ would be finite too. 2) If $g(y)$ is finite, then we have two subcases. 2.a) $g(y_1) = \infty$ and/or $g(y_2) = \infty$: in this case, the inequality is apparent; 2.b) $g(y_1)$ and $g(y_2)$ are finite: in this case, we have

$$g(y) = f_o(x) \leq \theta f_o(x_1) + (1-\theta) f_o(x_2) = \theta g(y_1) + (1-\theta) g(y_2),$$

which concludes the proof.
Finally, by infimizing over x we thus obtain a convex function.

4. (**2 points**) Consider the multiobjective optimization problem (with respect to cone $\mathbb{R}^2_+$) with objective function $f_o(x) = [F_1(x)\ F_2(x)]^T$ where $F_1(x) = x_1^2 + x_2^2$ and $F_2(x) = (2x_1+3)^2$.
a. Evaluate all the Pareto optimal values and points via scalarization (give explicit expressions for both values and points). I
b. Solve the scalarization problems with either weight equal to zero. For both cases, are the solutions of the scalar problem also Pareto optimal?

Sol.: Since the problem is convex, we know that all Pareto optimal points can be obtained via scalarization with some weight vector $\lambda \succeq 0$. We now fix some $\lambda \succ 0$ and solve the problem

$$\text{minimize } \lambda_1(x_1^2 + x_2^2) + \lambda_2(2x_1+3)^2,$$

which is equivalent to

$$\text{minimize } (\lambda_1 + 4\lambda_2)x_1^2 + \lambda_1 x_2^2 + 12\lambda_2 x_1 + 9\lambda_2.$$

Any solution to this problem will give us a Pareto optimal point and value. Since the cost function is stictly convex, the corresponding Pareto optimal point is given as

$$x^*(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{-6\lambda_2}{\lambda_1+4\lambda_2} \\ 0 \end{bmatrix},$$

which can be also restated in terms of $\mu = \lambda_2/\lambda_1$ as

$$x^*(\mu) = \begin{bmatrix} \frac{-6\mu}{1+4\mu} \\ 0 \end{bmatrix},$$

and the value is obtained by substituting the above into $f_o(x)$, which yields

$$\begin{aligned} F_1^*(\mu) &= \left(\frac{-6\mu}{1+4\mu}\right)^2 \\ F_2^*(\mu) &= \left(\frac{-12\mu}{1+4\mu} + 3\right)^2. \end{aligned}$$

To calculate the remaining Pareto optimal points and values, we need to let $\mu \to 0$ and $\mu \to \infty$. With $\mu \to 0$, we obtain $x^* = 0$ and $f_o^* = (0, 9)$, which corresponds to minimization of the norm only. This point can also be obtained with $\lambda_2 = 0$ and $\lambda_1 = 1$. With $\mu \to \infty$, we obtain $x^* = (-3/2, 0)^T$ and $f_o^* = (9/4, 0)$, which corresponds to minimizing the error with the solution with minimum norm. Note that if we solved the scalarization problem with $\lambda_2 = 1$ and $\lambda_1 = 0$, any point with $x_1 = -3/2$ is a solution, but is not necessarily a Pareto optimal point.

5. (2 points) Consider the game described by the payoff matrix below

$$\begin{array}{cc} 1,2 & 0,1 \\ 2,1 & 1,0 \end{array}$$

a. Identify the Pareto optimal points, and the Nash equilibria in pure strategies.
b. Calculate all Nash equilibria in mixed strategies.

Sol.: a. Pareto optimal values are (1,2) and (2,1), and the Nash equilibrium in pure strategy is (2,1).
b. Considering mixed strategies, the average utilities are

$$\begin{aligned} \bar{U}_1(p_1,p_2) &= p_1p_2 + (1-p_1)(2p_2 + (1-p_2)) \\ &= p_1p_2 + (1-p_1)(p_2+1) \\ &= -p_1 + (p_2+1) \\ \bar{U}_2(p_1,p_2) &= p_2(2p_1 + (1-p_1)) + (1-p_2)p_1 \\ &= p_2(p_1+1) + (1-p_2)p_1 \\ &= p_2 + p_1. \end{aligned}$$

This shows that player 1 always chooses $p_1 = 0$, irrespective of the action of player 2, and player 2 chooses always action $p_2 = 1$ irrespective of the action of player 1. This is also clear from the payoff matrix. Therefore, there are no mixed strategy equilibria, but only pure strategy equilibria.