ECE 673-Random signal analysis I
Final

Q1 (1 point) An amateur dart player hits with uniform probabilty a dartboard of radius $r = 10cm$ (see figure). Let us define X and Y as the coordinates of the point where the dart lands. What is the probability $P[X^2 + Y^2 = r_0^2]$ where $r_0 = 2cm$? What is the probability that the dart lands *within* a circle of radius r_0?
Sol.: The joint PDF of X and Y is clearly

$$p_{XY}(x, y) = \begin{cases} \frac{1}{\pi r^2} \text{ for } x^2 + y^2 \leq r^2 \\ 0 \text{ otherwise} \end{cases}$$

since we must have

$$\iint_{\mathcal{A}} p_{XY}(x, y)dxdy = 1,$$

where $\mathcal{A}$ denotes the dartboard.
Then, we have

$$P[X^2 + Y^2 = r_0^2] = 0,$$

since any line has zero area and therefore zero probability. However, the probability that the dart lands *within* a circle of radius r_0 is ($\mathcal{A}_0$ denotes the surface within the circle of radius r_0)

$$P[X^2 + Y^2 \leq r_0^2] = \iint_{\mathcal{A}_0} p_{XY}(x, y)dxdy = \frac{1}{\pi r^2}\pi r_0^2 = \frac{r_0^2}{r^2} = \frac{4}{100} = \frac{1}{25}.$$

Q2 (1 point) Comment on the following statement: "If $cov(X_1, X_2) = 0$, then the best predictor (without limitation on the type of predictor) of the random variable X_1 given the observation of X_2 is $E[X_1]$ and the corresponding prediction error is $var(X_1)$." Will your comment remain the same if you were told that the random variables X_1 and X_2 are bivariate Gaussian?
Sol.: If $cov(X_1, X_2) = 0$, i.e., if X_1 and X_2 are uncorrelated, then the best *linear* predictor is $E[X_1]$ and the corresponding prediction error is $var(X_1)$. However, if the random variables are uncorrelated but *not independent,* there could exist *non-linear* predictors with smaller prediction error than $var(X_1)$. On the other hand, if X_1 and X_2 are jointly Gaussian uncorrelation implies independence and $E[X_1]$ is in fact the optimum predictor.
Q3 (1 point) Two persons play a game in which the first thinks of a number (*any* number) from 0 to 1, while the second tries to guess player one's number. The second player claims that he is telepathic and knows what number the first player has chosen. In reality, he just choses a number at random (uniformly in the range between 0 and 1). If the first player also thinks of a number at random (i.e., uniformly distributed within 0 and 1), what is the probability that player two will choose a number whose difference from player one's number is less than 0.1?
Sol.: Let us denote as $X_1 \sim \mathcal{U}(0, 1)$ the number thought of by the first person and as $X_2 \sim \mathcal{U}(0, 1)$ the number selected by the "telepathic" second player. The probability we are interested in reads

$$P[X_1 - X_2 < 0.1].$$

But, since X_1 and X_2 are independent, the joint PDF is

$$p_{X_1X_2}(x_1,x_2) = \begin{cases} 1 \text{ for } 0 \leq x \leq 1 \ 0 \leq y \leq 1 \\ 0 \text{ otherwise} \end{cases}.$$

Therefore, the probability above can be obtained by simple geometrical considerations as ($\mathcal{A}=\{(x_1,x_2) : x_1 - x_2 < 0.1\}$):

$$\begin{aligned} P[X_1 - X_2 &< 0.1] = \iint_{\mathcal{A}} p_{X_1X_2}(x_1,x_2)dx_1dx_2 = \\ &= 1 - 0.9^2/2 = 0.595. \end{aligned}$$

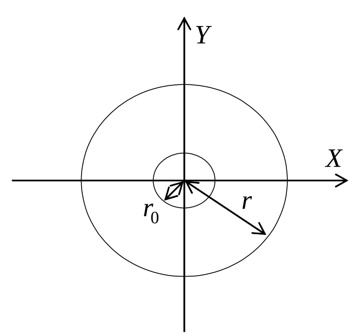
Q4 (1 point) Provide an example of a joint PDF $p_{X_1X_2}(x_1,x_2)$ such that $var(X_1+X_2) > var(X_1)+var(X_2)$. Then provide a second example where $var(X_1+X_2) < var(X_1)+var(X_2)$.
Sol.: We know that in general

$$var(X_1+X_2) = var(X_1) + var(X_2) + 2cov(X_1,X_2).$$

Therefore, in order to have $var(X_1+X_2) > var(X_1)+var(X_2)$, we must choose a joint PDF with positive covariance $cov(X_1,X_2)$, whereas the opposite holds for condition $var(X_1+X_2) < var(X_1)+var(X_2)$ (i.e., we must enforce $cov(X_1,X_2) < 0$). As an example, we could choose a bivariate Gaussian distribution

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

with either $\rho > 0$ ($cov(X_1,X_2) > 0$) or $\rho < 0$ ($cov(X_1,X_2) < 0$).



P1 (2 points) Two microphones are hidden in a room, where two conversations are taking place. The first microphone is closer to the first conversation and measures the signal X whereas the second microphone is set closer to second conversation and measures the signal Y. The random variables X and Y are bivariate Gassian (measured in Volt) with zero mean and power 1 $[V^2]$. Moreover, since the conversation recorded by one microphone is overheard also by the other, the two random variables are correlated with correlation 0.6.
a) Write the equation of the joint PDF of X and Y, $p_{XY}(x,y)$. Moreover, sketch $p_{XY}(x,y)$.
b) An alarm goes off every time either conversation become too heated, i.e., whenever $|X|$ or $|Y|$ is above $1.8V$. Assuming that $X = 1.6V$, what is the probability that the alarm goes off?

c) What is the probability that the signals recorded on the first and second microphones differ for more than $1V$, i.e.. $P[X-Y>1]$?
Sol.: a) X and Y are *standard* bivariate Gaussian:

$$\begin{aligned} p_{XY}(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right)= \\ &= \frac{1}{2\pi(0.8)}\exp\left(-\frac{x^2+y^2-1.2xy}{1.28}\right). \end{aligned}$$

See the textbook for sketches of standard bivariate Gaussian with positive correlation.
b) The probability we are interested in is

$$\begin{aligned} P[|Y| &> 1.8|X=1.6]= \\ &= P[Y>1.8|X=1.6]+P[Y<-1.8|X=1.6] \end{aligned}$$

But Y given the observation of X is a Gaussian random variable with

$$\begin{aligned} Y|(X &= 1.6)\sim\mathcal{N}(\rho\cdot 1.6, 1-\rho^2) \\ &\sim \mathcal{N}(0.96, 0.64) \end{aligned}$$

so that we can write

$$Y|(X=1.6)=0.96+\sqrt{0.64}Z$$

where Z is a standard Gaussian variable $Z\sim\mathcal{N}(0,1)$. Finally, the probability of interest can be calculated as

$$\begin{aligned} P[|Y| &> 1.8|X=1.6]=P\left[Z>\frac{1.8-0.96}{\sqrt{0.64}}\right]+P\left[Z<\frac{-1.8-0.96}{\sqrt{0.64}}\right]= \\ &= Q\left(\frac{1.8-0.96}{\sqrt{0.64}}\right)+\left(1-Q\left(\frac{-1.8-0.96}{\sqrt{0.64}}\right)\right)= \\ &= 0.147+0=0.147 \end{aligned}$$

as expected the probability $P[Y<-1.8|X=1.6]$ is negligible.
c) We need to evaluate $P[X-Y>1]$. But $T=X-Y$ is a Gaussian random variable with mean $E[T]=E[X-Y]=0$ and variance

$$\begin{aligned} var(T) &= var(X)+var(-Y)+2cov(X,-Y)= \\ &= 1+1-2cov(X,Y)=2-1.2=0.8. \end{aligned}$$

Therefore, we can express T as ($Z\sim\mathcal{N}(0,1)$)

$$T=\sqrt{0.8}Z$$

and finally

$$P[X-Y>1]=P[T>1]=Q\left(\frac{1}{\sqrt{0.8}}\right)=0.13.$$

P2 (2 points) A random walk process is defined as

$$Z[n]=\sum_{i=0}^{n}W[i],$$

where $W[n]$ is a white Gaussian noise with zero mean and power 1. It is well known that the random walk process $Z[n]$ is non-stationary.
a) The random walk process $Z[n]$ is input to a linear shift invariant (LSI) filter with system function $\mathcal{H}(z) = 1 - z^{-1}$. Show that the output $U[n]$ is stationary (Hint: write the difference equation).
Having defined $U[n]$ as the output of the LSI system at point a) (i.e., $U[n] = Z[n] - Z[n-1]$), consider now the random process $X[n]$:

$$X[n] = U[n] - 0.6U[n-1].$$

b) Is $X[n]$ wide sense stationary? Is it stationary?
c) Evaluate the autocorrelation sequence of $X[n]$ ($r_X[k]$). What is the power of $X[n]$? Do you expect the process to have a large power content at small or large frequencies?
d) Verify your conjecture at the previous point by evaluating and plotting the power spectral density $P_X(f)$.
Sol.: a) The output $U[n]$ can be written in terms of the difference equation as

$$\begin{aligned}
\frac{\mathcal{U}(z)}{\mathcal{Z}(z)} &= \mathcal{H}(z) \rightarrow \mathcal{U}(z) = \mathcal{Z}(z) - z^{-1}\mathcal{Z}(z) \\
&\rightarrow U[n] = Z[n] - Z[n-1] = \sum_{i=0}^{n} W[n] - \sum_{i=0}^{n-1} W[n] = W[n],
\end{aligned}$$

therefore we can conclude that $U[n] = W[n]$ is a white Gaussian noise with zero mean and power 1. The latter is not only wide sense stationary but also stationary for the property of Gaussian processes.
b) $X[n]$ is stationary (and thus also wide sense stationary) because it is the output of a LSI system with a white Gaussian noise at its input.
c) The autocorrelation sequence of $X[n]$ is

$$r_X[k] = r_h[k] = \sum_n h[n]h[n+k],$$

where the impulse response $h[n] = \delta[n] - 0.6\delta[n-1]$. Therefore, we have

$$r_X[k] = \begin{cases} 1.36 & \text{for } k = 0 \\ -0.6 & \text{for } k = \pm 1 \\ 0 & \text{for } |k| > 1 \end{cases}.$$

The power is $r_X[0] = 1.36$. Since the correlation between adjacent samples is negative, it is expected that the process present a more relevant power contribution at large frequencies.
d) The power spectral density reads

$$P_X(f) = \sum_k r_X[k] \exp(-j2\pi f k) = 1.36 - 1.2\cos(2\pi f).$$

The plot can be found in the course notes.
P3 (2 points) A random process is described by

$$X[n] = 0.5X[n-1] + U[n],$$

where $U[n]$ is a white Gaussian noise with zero mean and power 1.
a) Write the MATLAB code that generates and plots a realization of 1000 samples of $X[n]$.
b) The random process $X[n]$ is input to a LSI filter with system function $\mathcal{H}(z) = 1 - 0.5z^{-1}$ to generate the output random process $Y[n]$. Find the probability $P[Y[3] + Y[4] > 1]$ (Hint: consider $X[n]$ as the output of an LSI system).
Sol.:
a)

```
N=1000;
u=randn(N,1);
for n=1:N
 if (n==1) x(n)=u(n);
 else x(n)=u(n)+0.5*x(n-1);
  end
end
```

b) The random process $X[n]$ is a Gaussian process obtained by passing the white Gaussian noise $U[n]$ through a LSI system:

$$\begin{aligned} \mathcal{X}(z) &= 0.5z^{-1}\mathcal{X}(z) + \mathcal{U}(z) \\ &\rightarrow \mathcal{X}(z) = \frac{\mathcal{U}(z)}{1 - 0.5z^{-1}}. \end{aligned}$$

The process $Y[n]$ is too a stationary Gaussian process since it is the output of a LSI system fed by a stationary Gaussian process. Moreover, it reads

$$\mathcal{Y}(z) = \mathcal{X}(z)\left(1 - 0.5z^{-1}\right).$$

But

$$\mathcal{Y}(z) = \frac{\mathcal{U}(z)}{1 - 0.5z^{-1}}\left(1 - 0.5z^{-1}\right) = \mathcal{U}(z).$$

In conclusion $Y[n] = U[n]$, that is $U[n]$ is a white Gaussian noise with zero mean and power 1. Therefore, $Y[3] + Y[4]$ is $\mathcal{N}(0, 2)$ and the requested probability is

$$P[Y[3] + Y[4] > 1] = Q(1/\sqrt{2}) = 0.24$$