

Heegard–Berger and Cascade Source Coding Problems With Common Reconstruction Constraints

Behzad Ahmadi, *Student Member, IEEE*, Ravi Tandon, *Member, IEEE*, Osvaldo Simeone, *Member, IEEE*, and H. Vincent Poor, *Fellow, IEEE*

Abstract—In lossy source coding with side information at the decoder (i.e., the Wyner–Ziv problem), the estimate of the source obtained at the decoder cannot be generally reproduced at the encoder, due to its dependence on the side information. In some applications, this may be undesirable, and a common reconstruction (CR) requirement, whereby one imposes that the encoder and decoder be able to agree on the decoder’s estimate, may be instead in order. The rate-distortion function under the CR constraint has been derived recently for a point-to-point (Wyner–Ziv) problem. In this paper, this result is extended to three multiterminal settings with three nodes, namely the Heegard–Berger (HB) problem, its variant with cooperating decoders, and the cascade source coding problem. The HB problem consists of an encoder broadcasting to two decoders with respective side information. The cascade source coding problem is characterized by a two-hop system with side information available at the intermediate and final nodes. For the HB problem with the CR constraint, the rate-distortion function is derived under the assumption that the side information sequences are (stochastically) degraded. The rate-distortion function is also calculated explicitly for three examples, namely Gaussian source and side information with quadratic distortion metric, and binary source and side information with erasure and Hamming distortion metrics. The rate-distortion function is then characterized for the HB problem with cooperating decoders and (physically) degraded side information. For the cascade problem with the CR constraint, the rate-distortion region is obtained under the assumption that side information at the final node is physically degraded with respect to that at the intermediate node. For the latter two cases, it is worth emphasizing that the corresponding problem without the CR constraint is still open. Outer and inner bounds on the rate-distortion region are also obtained for the cascade problem under the assumption that the side information at the intermediate node is physically degraded with respect to that at the final node. For the three examples mentioned above, the bounds are shown to coincide. Finally, for the HB problem, the rate-distortion function is obtained under the more general requirement of constrained reconstruction, whereby the decoder’s estimate must be recovered at the encoder only within some distortion.

Index Terms—Cascade source coding, common reconstruction (CR), Heegard–Berger (HB) problem, source coding with side information.

I. INTRODUCTION

SOURCE coding problems with side information at the decoder(s) model a large number of scenarios of practical interest, including video streaming [1] and wireless sensor networks [2]. From an information-theoretic perspective, the baseline setting for this class of problems is one in which a memoryless source $X^n = (X_1, \dots, X_n)$ is to be communicated by an encoder at a rate R bits per source symbol to a decoder that has available a correlated sequence Y^n that is related to X^n via a memoryless channel $p(y|x)$ (see Fig. 1¹). Under the requirement of asymptotically lossless reconstruction \hat{X}^n of the source X^n at the decoder, the minimum required rate was obtained by Slepian and Wolf [3]. Later, the more general optimal tradeoff between rate R and the distortion D between the source X^n and reconstruction \hat{X}^n was obtained by Wyner and Ziv [4] for any given distortion metric $d(x, \hat{x})$. It was shown to be given by the rate-distortion function

$$R_{X|Y}^{\text{WZ}}(D) = \min I(X; U|Y) \quad (1)$$

where the minimum is taken over all probability mass functions (pmfs) $p(u|x)$ and deterministic function $\hat{x}(u, y)$ such that $E[d(X, \hat{x}(U, Y))] \leq D$.

A. Heegard–Berger and Cascade Source Coding Problems

In applications such as the ones discussed above, the point-to-point setting of Fig. 1 does not fully capture the main features of the source coding problem. For instance, in video streaming, a transmitter typically *broadcasts* information to a number of decoders. As another example, in sensor networks, data are typically routed over *multiple hops* toward the destination. A model that accounts for the aspect of *broadcasting* to multiple decoders is the *Heegard–Berger (HB)* setup shown in Fig. 2. In this model, the link of rate R bits per source symbol is used to communicate to two receivers having different side information sequences, Y_1^n and Y_2^n , which are related to the source X^n via a memoryless channel $p(y_1, y_2|x)$. The set of the achievable triples (R, D_1, D_2) for this model, where D_1 and D_2 are the distortion levels at Decoders 1 and 2, respectively, was derived in [6] and [7] under the assumption that the side information sequences are (stochastically) degraded versions of the source X^n . In a variation of this model shown in Fig. 3, decoder cooperation is enabled by a limited capacity link from one decoder (Decoder 1) to the other (Decoder 2). Inner and outer bounds on the rate-distortion region for this problem are obtained in [8] under the assumption that the side information of Decoder 2 is (physically) degraded with respect to that of Decoder 1.

As for *multihopping*, a basic model that captures some of the key design issues is shown in Fig. 4. In this *cascade* setup, an

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B. Ahmadi and O. Simeone are with the Center of Wireless Communication and Signal Processing Research, New Jersey Institute of Technology, Newark, NJ 07102 USA (e-mail: behzad.ahmadi@njit.edu; osvaldo.simeone@njit.edu).

R. Tandon was with Princeton University, Princeton, NJ 08544 USA. He is now with Virginia Tech, Blacksburg, VA 24061 USA (e-mail: tandonr@vt.edu; rtandon@princeton.edu).

H. V. Poor is with Princeton University, Princeton, NJ 08544 USA (e-mail: poor@princeton.edu).

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¹The presence of the function ψ at the encoder will be explained later.

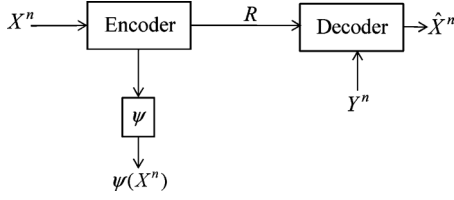


Fig. 1. Point-to-point source coding with common reconstruction [5].

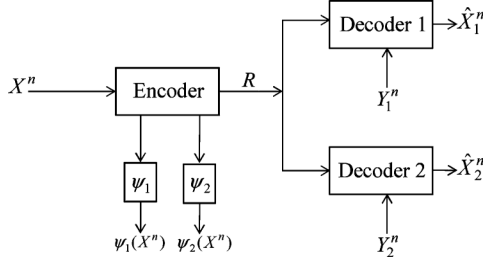


Fig. 2. HB source coding problem with common reconstruction.

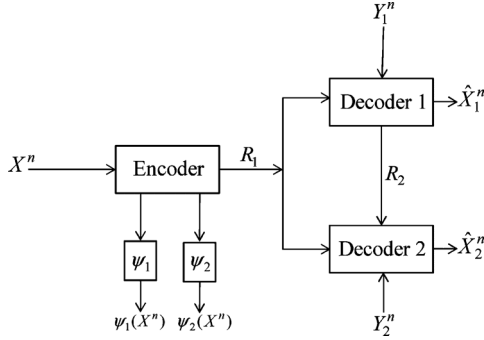


Fig. 3. HB source coding problem with common reconstruction and decoder cooperation.

encoder (Node 1) communicates with rate R_1 to an intermediate node (Node 2), which has side information Y_1^n , and in turn communicates with rate R_2 to a final node (Node 3) with side information Y_2^n . Both Nodes 2 and 3 act as decoders, similar to the HB problem of Fig. 2, in the sense that they reconstruct a local estimate of the source X^n . The rate-distortion function for this problem has been derived for various special cases in [9]–[11] and [12] (see [12, Table I] for an overview). Chia *et al.* [11] derive the set of all achievable quadruples (R_1, R_2, D_1, D_2) , i.e., the rate-distortion region, for the case in which Y_1^n is also available at the encoder and Y_2^n is a physically degraded version of X^n with respect to Y_1^n . Instead, Vasudevan *et al.* [10] derive the rate-distortion region under the assumptions that the source and the side information sequences are jointly Gaussian, that the distortion metric is quadratic, and that the sequence Y_1^n is a physically degraded version of X^n with respect to Y_2^n . The corresponding result for binary source and side information and Hamming distortion metric was derived in [12].

B. Common Reconstruction Constraint

A key aspect of the optimal strategies identified in [4], [6], [7], [10], and [11] is that the side information sequences are, in general, used in two different ways: (i) as a means to reduce

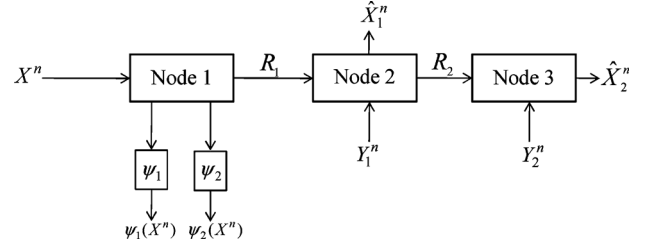


Fig. 4. Cascade source coding problem with common reconstruction.

the rate required for communication between encoder and decoders via binning; and (ii) as an additional observation that the decoder can leverage, along with the bits received from the encoder, in order to improve its local estimate. For instance, for the point-to-point system of Fig. 1, the Wyner–Ziv result (1) reflects point (i) of the discussion above in the conditioning on side information Y , which reduces the rate, and point (ii) in the fact that the reconstruction \hat{X} is a function $\hat{x}(U, Y)$ of the signal U received from the encoder and the side information Y .

Leveraging the side information as per point (ii), while advantageous in terms of rate-distortion tradeoff, may have unacceptable consequences for some applications. In fact, this use of side information entails that the reconstruction \hat{X} of the decoder cannot be reproduced at the encoder. In other words, the encoder and decoder cannot agree on the specific reconstruction \hat{X} obtained at the receiver side, but only on the average distortion level D . In applications such as transmission of sensitive medical, military or financial data, this may not be desirable. Instead, one may want to add the constraint that the reconstruction at the decoder be reproducible by the encoder [5]. This idea, referred to as the common reconstruction (CR) constraint, was first proposed in [5], where it is shown for the point-to-point setting of Fig. 1² that the rate-distortion function under the CR constraint is given by

$$R_{X|Y}^{CR}(D) = \min I(X; \hat{X}|Y) \quad (2)$$

where the minimum is taken over all pmfs $p(\hat{x}|x)$ such that $E[d(X, \hat{X})] \leq D$. Comparing (2) with the Wyner–Ziv rate-distortion (1), it can be seen that the additional CR constraint prevents the decoder from using the side information as a means to improve its estimate \hat{X} (see point (ii) above).

The original work of Steinberg [5] has been recently extended in [13], where a relaxed CR constraint is imposed in which only a distortion constraint is imposed between the decoder's reconstruction and its reproduction at the encoder. We refer to this setting as imposing a *constrained reconstruction* (ConR) requirement.

C. Main Contributions

In this paper, we study the HB source coding problem (see Fig. 2) and the cascade source coding problem (see Fig. 4) under the CR requirement. The considered models are thus relevant for the transmission of sensitive information, which is constrained by CR, via broadcast or multihop links—a common occurrence

²The function ψ at the encoder calculates the estimate of the encoder regarding the decoder's reconstruction.

in, e.g., medical, military, or financial applications (e.g., for intranets of hospitals or financial institutions). Specifically, our main contributions are the following.

- 1) For the HB problem with the CR constraint (see Fig. 2), we derive the rate-distortion function under the assumption that the side information sequences are (stochastically) degraded. We also calculate this function explicitly for three examples, namely Gaussian source and side information with quadratic distortion metric, and binary source and erasure side information with erasure and Hamming distortion metrics (see Section II).
- 2) For the HB problem with the CR constraint and decoder cooperation (see Fig. 3), we derive the rate-distortion region under the assumption that the side information sequences are (physically) degraded in either direction (see Sections III-A and III-B). We emphasize that the corresponding problem without the CR constraint is still open as per the discussion above.
- 3) For the cascade problem with the CR constraint (see Fig. 4), we obtain the rate-distortion region under the assumption that side information Y_2 is physically degraded with respect to Y_1 (see Section IV-B). We emphasize that the corresponding problem without the CR constraint is still open as per the discussion above.
- 4) For the cascade problem with CR constraint (see Fig. 4), we obtain outer and inner bounds on the rate-distortion region under the assumption that the side information Y_1 is physically degraded with respect to Y_2 . Moreover, for the three examples mentioned above in the context of the HB problem, we show that the bounds coincide and we evaluate the corresponding rate-distortion region explicitly (see Section IV-C).
- 5) For the HB problem, we finally derive the rate-distortion function under the more general requirement of ConR and under the assumption that side information Y_2 is physically degraded with respect to Y_1 (Sec.V).

Notation: For a and b integer with $a \leq b$, we define $[a, b]$ as the interval $[a, a + 1, \dots, b]$ and we use x_a^b to denote the sequence (x_a, \dots, x_b) . We will also write x^b for x_1^b for simplicity. Upper case, lower case, and calligraphic letters denote random variables, specific values of random variables and their alphabets, respectively. Given discrete random variables, or more generally vectors, X and Y , we will use the notation $p_X(x)$ or $p(x)$ for $\Pr[X = x]$, and $p_{X|Y}(x|y)$ or $p(x|y)$ for $\Pr[X = x|Y = y]$, where the latter notations are used when the meaning is clear from the context. Given a set \mathcal{X} , we denoted by \mathcal{X}^n the n -fold Cartesian product of \mathcal{X} . For random variables X and Y , we denote by $\sigma_{X|Y}^2$ the (average) conditional variance of X given Y , i.e., $\mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y]]$. We adopt the notation convention in [14], in which $\delta(\epsilon)$ represents any function such that $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. We define the binary entropy function $H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$. Finally, we define $\alpha * \beta = \alpha(1-\beta) + \beta(1-\alpha)$.

II. HB PROBLEM WITH CR

In this section, we first detail the system model for the HB source coding problem in Fig. 2 with CR in Section II-A. Next, the characterization of the corresponding rate-distortion per-

formance is derived under the assumption that one of the two side information sequences is a stochastically degraded version of the other in the sense of [6] [see (10)]. Finally, three specific examples are worked out, namely Gaussian sources under quadratic distortion (see Section II-C), and binary sources with side information sequences subject to erasures under Hamming or erasure distortion (see Section II-D).

A. System Model

In this section, the system model for the HB problem with CR is detailed. The system is defined by the pmf $p_{XY_1Y_2}(x, y_1, y_2)$ and discrete alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, \hat{\mathcal{X}}_1$, and $\hat{\mathcal{X}}_2$ as follows. The source sequence X^n and side information sequences Y_1^n and Y_2^n , with $X^n \in \mathcal{X}^n, Y_1^n \in \mathcal{Y}_1^n$, and $Y_2^n \in \mathcal{Y}_2^n$, are such that the tuples (X_i, Y_{1i}, Y_{2i}) for $i \in [1, n]$ are independent and identically distributed (i.i.d.) with joint pmf $p_{XY_1Y_2}(x, y_1, y_2)$. The encoder measures a sequence X^n and encodes it into a message J of nR bits, which is delivered to the decoders. Decoders 1 and 2 wish to reconstruct the source sequence X^n within given distortion requirements, to be discussed below, as $\hat{X}_1^n \in \hat{\mathcal{X}}_1^n$ and $\hat{X}_2^n \in \hat{\mathcal{X}}_2^n$, respectively. The estimated sequence \hat{X}_j^n is obtained as a function of the message J and the side information sequence Y_j^n for $j = 1, 2$. The estimates are constrained to satisfy distortion constraints defined by per-symbol distortion metrics $d_j(x, \hat{x}_j) : \mathcal{X} \times \hat{\mathcal{X}}_j \rightarrow [0, D_{\max}]$ with $0 < D_{\max} < \infty$. Based on the given distortion metrics, the overall distortion for the estimated sequences \hat{x}_1^n and \hat{x}_2^n is defined as

$$d_j^n(x^n, \hat{x}_j^n) = \frac{1}{n} \sum_{i=1}^n d_j(x_i, \hat{x}_{ji}), \quad \text{for } j = 1, 2. \quad (3)$$

The reconstructions \hat{X}_2^n and \hat{X}_1^n are also required to satisfy the CR constraints, as formalized below.

Definition 1: An $(n, R, D_1, D_2, \epsilon)$ code for the HB problem with CR consists of an encoding function

$$g: \mathcal{X}^n \rightarrow [1, 2^{nR}] \quad (4)$$

which maps the source sequence X^n into a message J ; a decoding function for Decoder 1

$$h_1: [1, 2^{nR}] \times \mathcal{Y}_1^n \rightarrow \hat{\mathcal{X}}_1^n \quad (5)$$

which maps the message J and the side information Y_1^n into the estimated sequence \hat{X}_1^n ; a decoding function for Decoder 2

$$h_2: [1, 2^{nR}] \times \mathcal{Y}_2^n \rightarrow \hat{\mathcal{X}}_2^n \quad (6)$$

which maps message J and the side information Y_2^n into the estimated sequence \hat{X}_2^n ; and two reconstruction functions

$$\psi_1: \mathcal{X}^n \rightarrow \hat{\mathcal{X}}_1^n \quad (7a)$$

$$\text{and } \psi_2: \mathcal{X}^n \rightarrow \hat{\mathcal{X}}_2^n \quad (7b)$$

which map the source sequence into the estimated sequences at the encoder, namely $\psi_1(X^n)$ and $\psi_2(X^n)$, respectively, such that the distortion constraints are satisfied, i.e.,

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} [d_j(X_i, \hat{X}_{ji})] \leq D_j, \quad \text{for } j = 1, 2 \quad (8)$$

and the CR requirements hold, namely

$$\Pr \left[\psi_j(X^n) \neq \hat{X}_j^n \right] \leq \epsilon, \quad j = 1, 2. \quad (9)$$

Given distortion pairs (D_1, D_2) , a rate pair R is said to be achievable if, for any $\epsilon > 0$ and sufficiently large n , there exists an $(n, R, D_1 + \epsilon, D_2 + \epsilon, \epsilon)$ code. The rate-distortion function $R(D_1, D_2)$ is defined as $R(D_1, D_2) = \inf \{R : \text{the triple } (R, D_1, D_2) \text{ is achievable}\}$.

B. Rate-Distortion Function

In this section, a single-letter characterization of the rate-distortion function for the HB problem with CR is derived, under the assumption that the joint pmf $p(x, y_1, y_2)$ is such that there exists a conditional pmf $\tilde{p}(y_1|y_2)$ for which

$$p(x, y_1) = \sum_{y_2 \in \mathcal{Y}_2} p(x, y_2) \tilde{p}(y_1|y_2). \quad (10)$$

In other words, the side information Y_1 is a stochastically degraded version of Y_2 .

Proposition 1: If the side information Y_1 is stochastically degraded with respect to Y_2 , the rate-distortion function for the HB problem with CR is given by

$$R_{HB}^{CR}(D_1, D_2) = \min I(X; \hat{X}_1|Y_1) + I(X; \hat{X}_2|Y_2\hat{X}_1) \quad (11)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, \hat{x}_1, \hat{x}_2) = p(x, y_1, y_2) p(\hat{x}_1, \hat{x}_2|x) \quad (12)$$

and minimization is performed with respect to the conditional pmf $p(\hat{x}_1, \hat{x}_2|x)$ under the constraints

$$\mathbb{E}[d_j(X, \hat{X}_j)] \leq D_j, \quad \text{for } j = 1, 2. \quad (13)$$

The proof of the converse can be found in Appendix A. Achievability follows as a special case of [6, Th. 3] and can be easily shown using standard arguments. In particular, the encoder randomly generates a standard lossy source code \hat{X}_1^n for the source X^n with rate $I(X; \hat{X}_1)$ bits per source symbol. Random binning is used to reduce the rate to $I(X; \hat{X}_1|Y_1)$. By the Wyner–Ziv theorem [14, p. 280], this guarantees that both Decoders 1 and 2 are able to recover \hat{X}_1^n (since Y_1 is a degraded version of Y_2). The encoder then maps the source X^n into the reconstruction sequence \hat{X}_2^n using a codebook that is generated conditional on \hat{X}_1^n with rate $I(X; \hat{X}_2|\hat{X}_1)$ bits per source symbol. Random binning is again used to reduce the rate to $I(X; \hat{X}_2|Y_2\hat{X}_1)$. From the Wyner–Ziv theorem, and the fact that Decoder 2 knows the sequence \hat{X}_1^n , it follows that Decoder 2 can recover the reconstruction \hat{X}_2^n as well. Note that, since the reconstruction sequences \hat{X}_1^n and \hat{X}_2^n are generated by the encoder, functions ψ_1 and ψ_2 that guarantee the CR constraints (9) exist by construction.

Remark 1: Under the physical degradedness assumption that the Markov chain condition $X \rightarrow Y_2 \rightarrow Y_1$ holds, (11) can be rewritten as

$$R = \min I(X; \hat{X}_1\hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \quad (14)$$

with the minimization defined as in (11). This expression quantifies by $I(\hat{X}_1; Y_2|Y_1)$ the additional rate that is required with respect to the ideal case in which both decoders have the better side information Y_2 .

Remark 2: If we remove the CR constraint, then the rate-distortion function under the assumption of Proposition 1 is given by [6]

$$R_{HB}(D_1, D_2) = \min I(X; U_1|Y_1) + I(X; U_2|Y_2U_1) \quad (15)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$\begin{aligned} p(x, y_1, y_2, u_1, u_2, \hat{x}_1, \hat{x}_2) \\ = p(x, y_1, y_2) p(u_1, u_2|x) \delta(\hat{x}_1 - \hat{x}_1(u_1, y_1)) \delta(\hat{x}_2 - \hat{x}_2(u_2, y_2)) \end{aligned} \quad (16)$$

and minimization is performed with respect to the conditional pmf $p(u_1, u_2|x)$ and the deterministic functions $\hat{x}_j(u_j, y_j)$, for $j = 1, 2$, such that distortion constraints (13) are satisfied. Comparison of (11) with (15) reveals that, similar to the discussion around (1) and (2), the CR constraint permits the use of side information only to reduce the rate via binning, but not to improve the decoder's estimates via the use of the auxiliary codebooks represented by variables U_1 and U_2 , and functions $\hat{x}_j(u_j, y_j)$, for $j = 1, 2$, in (16).

Remark 3: Consider the case in which the side information sequences are available in a causal fashion in the sense of [16], that is, the decoding functions (5)–(6) are modified as $h_{ji}: [1, 2^{nR}] \times \mathcal{Y}_j^i \rightarrow \hat{\mathcal{X}}_{ji}$, for $i \in [1, n]$ and $j = 1, 2$, respectively. Following similar steps as in the proof of Proposition 2 and in [16], it can be concluded that, under the CR constraint, the rate-distortion function in this case is the same as if the two side information sequences were not available at the decoders and is thus given by (11) upon removing the conditioning on the side information. Note that this is true irrespective of the joint pmf $p(x, y_1, y_2)$, and hence, it holds also for nondegraded side information. This result can be explained by noting that, as explained in [16], causal side information prevents the possibility of reducing the rate via binning. Since the CR constraint also prevents the side information from being used to improve the decoders' estimates, it follows that the side information is useless in terms of rate-distortion performance, if used causally under the CR constraint.

On a similar note, if only side information Y_1 is causally available, while Y_2 can still be used in the conventional non-causal fashion, then it can be proved that Y_1 can be neglected without loss of optimality. Therefore, the rate-distortion function follows from (11) by removing the conditioning on Y_1 .

Remark 4: In [19], a related model is studied in which the source is given as $X = (Y_1, Y_2)$ and each decoder is interested

in reconstructing a lossy version of the side information available at the other decoder. The CR constraint is imposed in a different way by requiring that each decoder be able to reproduce the estimate reconstructed at the other decoder.

C. Gaussian Sources and Quadratic Distortion

In this section, we highlight the result of Proposition 1 by considering a zero-mean Gaussian source $X \sim \mathcal{N}(0, \sigma_x^2)$, with side information variables

$$Y_1 = X + Z_1 \quad (17a)$$

$$\text{and } Y_2 = X + Z_2 \quad (17b)$$

where $Z_1 \sim \mathcal{N}(0, N_1 + N_2)$ and $Z_2 \sim \mathcal{N}(0, N_2)$ are independent of each other and of Y_2 and X . Note that the joint distribution of (X, Y_1, Y_2) satisfies the stochastic degradedness condition. We focus on the quadratic distortion $d_j(x, \hat{x}_j) = (x - \hat{x}_j)^2$ for $j = 1, 2$. By leveraging standard arguments that allow us to apply Proposition 1 to Gaussian sources under mean-square-error constraint (see [14, pp. 50–51] and [15]), we obtain a characterization of the rate-distortion function for the given distortion and metrics.

We first recall that for the point-to-point setup in Fig. 1 with $X \sim \mathcal{N}(0, \sigma_x^2)$ and side information $Y = X + Z$, with $Z \sim \mathcal{N}(0, N)$ independent of X , the rate-distortion function with CR under quadratic distortion is given by [5]

$$R_{X|Y}^{\text{CR}}(D) = \begin{cases} R_G^{\text{CR}}(D, N) \triangleq \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{\sigma_x^2 + N} \cdot \frac{D + N}{D} \right), & \text{for } D \leq \sigma_x^2 \\ 0, & \text{for } D > \sigma_x^2 \end{cases} \quad (18)$$

where we have made explicit dependence on N of the function $R_G^{\text{CR}}(D, N)$ for convenience. The rate-distortion function (18) for $D \leq \sigma_x^2$ is obtained from (2) by choosing the distribution $p(\hat{x}|x)$ such that $X = \hat{X} + Q$, where $Q \sim \mathcal{N}(0, D)$ is independent of \hat{X} .

Proposition 2: The rate-distortion function for the HB problem with CR for Gaussian sources (17) and quadratic distortion is given by

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) = \begin{cases} 0, & \text{if } D_1 \geq \sigma_x^2 \text{ and } D_2 \geq \sigma_x^2 \\ R_G^{\text{CR}}(D_1, N_1 + N_2), & \text{if } D_1 \leq \sigma_x^2 \text{ and } D_2 \geq \min(D_1, \sigma_x^2) \\ R_G^{\text{CR}}(D_2, N_2), & \text{if } D_1 \geq \sigma_x^2 \text{ and } D_2 \leq \sigma_x^2 \\ \tilde{R}_{\text{HB}}^{\text{CR}}(D_1, D_2), & \text{if } D_2 \leq D_1 \leq \sigma_x^2 \end{cases} \quad (19)$$

where $R_G^{\text{CR}}(D, N)$ is defined in (18) and

$$\tilde{R}_{\text{HB}}^{\text{CR}}(D_1, D_2) \triangleq \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{(\sigma_x^2 + N_1 + N_2)} \cdot \frac{(D_1 + N_1 + N_2)(D_2 + N_2)}{(D_1 + N_2)D_2} \right). \quad (20)$$

Remark 5: The rate-distortion function for the HB problem for Gaussian sources (17) without the CR constraint can be

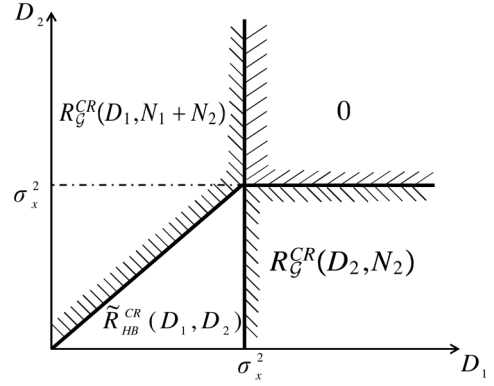


Fig. 5. Illustration of the distortion regions in the rate-distortion function (19) for Gaussian sources and quadratic distortion.

found in [6]. Comparison with (19) confirms the performance loss discussed in Remark 2.

Definition of the rate-distortion function (19) requires different consideration for the four subregions of the (D_1, D_2) plane sketched in Fig. 5. In fact, for $D_1 \geq \sigma_x^2$ and $D_2 \geq \sigma_x^2$, the required rate is zero, since the distortion constraints are trivially met by setting $\hat{X}_1 = \hat{X}_2 = 0$ in the achievable rate (11). For the case $D_1 \geq \sigma_x^2$ and $D_2 \leq \sigma_x^2$, it is sufficient to cater only to Decoder 2 by setting $\hat{X}_1 = 0$ and $X = \hat{X}_2 + Q_2$, with $Q_2 \sim \mathcal{N}(0, D_2)$ independent of \hat{X}_2 , in the achievable rate (11). That this rate cannot be improved upon follows from the trivial converse

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) \geq \max\{R_G^{\text{CR}}(D_1, N_1 + N_2), R_G^{\text{CR}}(D_2, N_2)\} \quad (21)$$

which follows by cut-set arguments. The same converse suffices also for the regime $D_1 \leq \sigma_x^2$ and $D_2 \geq \min(D_1, \sigma_x^2)$. For this case, achievability follows by setting $X = \hat{X}_1 + Q_1$ and $\hat{X}_1 = \hat{X}_2$ in (11), where $Q_1 \sim \mathcal{N}(0, D_1)$ is independent of \hat{X}_1 . In the remaining case, namely $D_2 \leq D_1 \leq \sigma_x^2$, the rate-distortion function does not follow from the point-to-point result (18) as for the regimes discussed thus far. The analysis of this case requires use of the entropy-power inequality (EPI) and can be found in Appendix B

Fig. 6 depicts the rate $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ in (19) versus D_1 for different values of D_2 with $\sigma_x^2 = 4$, $N_1 = 2$, and $N_2 = 3$. As discussed above, for $D_2 = 5$, which is larger than σ_x^2 , $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ becomes zero for values of D_1 larger than $\sigma_x^2 = 4$, while this is not the case for values $D_2 < \sigma_x^2 = 4$.

D. Binary Source With Erased Side Information and Hamming or Erasure Distortion

In this section, we consider a binary source $X \sim \text{Ber}(\frac{1}{2})$ with erased side information sequences Y_1 and Y_2 . The source Y_2 is an erased version of the source X with erasure probability p_2 and Y_1 is an erased version of X with erasure probability $p_1 > p_2$. This means that $Y_j = e$, where e represents an erasure, with probability p_j and $Y_j = X$ with probability $1 - p_j$. Note that, with these assumptions, the side information Y_1 is stochastically degraded with respect to Y_2 . In fact, we have the factorization (10), where additional distributions $p(y_2|x)$ and $\tilde{p}(y_1|y_2)$ are illustrated in Fig. 7. As seen in Fig. 7,

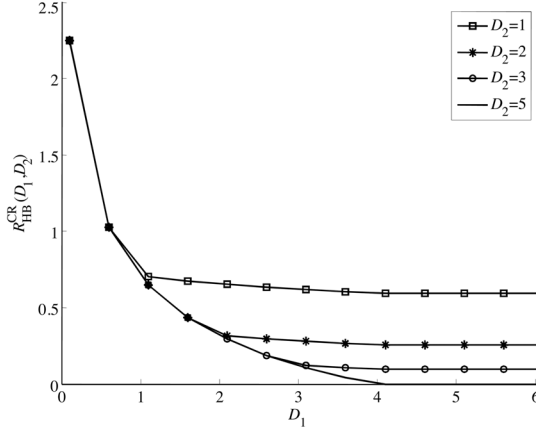


Fig. 6. Rate-distortion function $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ in (19) versus distortion D_1 for different values of distortion D_2 and for $\sigma_x^2 = 4$, $N_1 = 2$, and $N_2 = 3$.

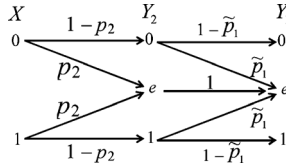


Fig. 7. Illustration of the pmfs in the factorization (10) of the joint distribution $p(x, y_1, y_2)$ for a binary source X and erased side information sequences (Y_1, Y_2) .

the pmf $\tilde{p}(y_1|y_2)$ is characterized by the probability \tilde{p}_1 that satisfies the equality $p_1 = p_2 + \tilde{p}_1(1 - p_2)$. We focus on Hamming and erasure distortions. For Hamming distortion, the reconstruction alphabets are binary, $\hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2 = \{0, 1\}$, and we have $d_j(x, \hat{x}_j) = 0$ if $x = \hat{x}_j$ and $d_j(x, \hat{x}_j) = 1$ otherwise for $j = 1, 2$. Instead, for the erasure distortion, the reconstruction alphabets are $\hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2 = \{0, 1, e\}$, and we have for $j = 1, 2$:

$$d_j(x, \hat{x}_j) = \begin{cases} 0, & \text{for } \hat{x}_j = x \\ 1, & \text{for } \hat{x}_j = e \\ \infty, & \text{otherwise.} \end{cases} \quad (22)$$

In Appendix C, we prove that for the point-to-point setup in Fig. 1 with $X \sim \text{Ber}(\frac{1}{2})$ and erased side information Y , with erasure probability p , the rate-distortion function with CR under Hamming distortion is given by

$$R_{X|Y}^{\text{CR}}(D) = \begin{cases} R_B^{\text{CR}}(D, p) \triangleq p(1 - H(D)), & \text{for } D \leq 1/2 \\ 0, & \text{for } D > 1/2 \end{cases} \quad (23)$$

where we have made explicit the dependence on p of the function $R_B^{\text{CR}}(D, p)$ for convenience. The rate-distortion function (23) for $D \leq 1/2$ is obtained from (2) by choosing the distribution $p(\hat{x}|x)$ such that $X = \hat{X} \oplus Q$, where $Q \sim \text{Ber}(D)$ is independent of \hat{X} . Following the same steps as in Appendix C, it can be also proved that for the point-to-point setup in Fig. 1 with $X \sim \text{Ber}(\frac{1}{2})$ and erased side information Y , with erasure probability p , the rate-distortion function with CR under erasure distortion is given by

$$R_{X|Y}^{\text{CR}}(D) = R_{B\mathcal{E}}^{\text{CR}}(D, p) \triangleq p(1 - D). \quad (24)$$

The rate-distortion function (24) is obtained from (2) by choosing the distribution $p(\hat{x}|x)$ such that $\hat{X} = X$ with probability $1 - D$ and $\hat{X} = e$ with probability D .

Remark 6: The rate-distortion function with erased side information and Hamming distortion without the CR constraint is derived in [17] (see also [18]). Comparison with (23) shows again the limitation imposed by the CR constraint on the use of side information (see Remark 2).

Proposition 3: The rate-distortion function for the HB problem with CR for the binary source with the stochastically degraded erased side information sequences illustrated in Fig. 7 under Hamming distortion is given by

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) = \begin{cases} 0, & \text{if } D_1 \geq 1/2 \text{ and } D_2 \geq 1/2 \\ R_B^{\text{CR}}(D_1, p_1), & \text{if } D_1 \leq 1/2 \text{ and } D_2 \geq \min(D_1, 1/2) \\ R_B^{\text{CR}}(D_2, p_2), & \text{if } D_1 \geq 1/2 \text{ and } D_2 \leq 1/2 \\ R_{\text{HB}}^{\text{CR}}(D_1, D_2), & \text{if } D_2 \leq D_1 \leq 1/2 \end{cases} \quad (25)$$

where $R_B^{\text{CR}}(D, N)$ is defined in (23) and

$$\tilde{R}_{\text{HB}}^{\text{CR}}(D_1, D_2) \triangleq p_1(1 - H(D_1)) + p_2(H(D_1) - H(D_2)). \quad (26)$$

Moreover, for the same source under erasure distortion, the rate-distortion function is given by (25) by substituting $R_B^{\text{CR}}(D_j, p_j)$ with $R_{B\mathcal{E}}^{\text{CR}}(D_j, p_j)$ as defined in (24) for $j = 1, 2$ and by substituting (26) with

$$\tilde{R}_{\text{HB},\mathcal{E}}^{\text{CR}}(D_1, D_2) \triangleq p_1(1 - D_1) + p_2(D_1 - D_2). \quad (27)$$

Similar to the Gaussian example, the characterization of the rate-distortion function (25) requires different considerations for the four subregions of the (D_1, D_2) plane sketched in Fig. 8. In fact, for $D_1 \geq 1/2$ and $D_2 \geq 1/2$, the required rate is zero, since the distortion constraints are trivially met by setting $\hat{X}_1 = \hat{X}_2 = 0$ in the achievable rate (11). For the case $D_1 \geq 1/2$ and $D_2 \leq 1/2$, it is sufficient to cater only to Decoder 2 by setting $\hat{X}_1 = 0$ and $X = \hat{X}_2 \oplus Q_2$, with $Q_2 \sim \text{Ber}(D_2)$ independent of X , in the achievable rate (11). That this rate cannot be improved upon is a consequence from the trivial converse

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) \geq \max\{R_B^{\text{CR}}(D_1, p_1), R_B^{\text{CR}}(D_2, p_2)\} \quad (28)$$

which follows by cut-set arguments. The same converse suffices also for the regime $D_1 \leq 1/2$ and $D_2 \geq \min(D_1, 1/2)$. For this case, achievability follows by setting $X = \hat{X}_1 \oplus Q_1$ and $\hat{X}_1 = \hat{X}_2$ in (11), where $Q_1 \sim \text{Ber}(D_1)$ is independent of \hat{X}_1 . In the remaining case, namely $D_2 \leq D_1 \leq 1/2$, the rate-distortion function does not follow from the point-to-point result (23) as for the regimes discussed thus far. The analysis of this case can be found in Appendix D. Similar arguments apply also for the erasure distortion metric.

We now compare the rate-distortion function for the binary source $X \sim \text{Ber}(\frac{1}{2})$ with erased side information under Hamming distortion for three settings. In the first setting, known as the Kaspi model [7], the encoder knows the side information

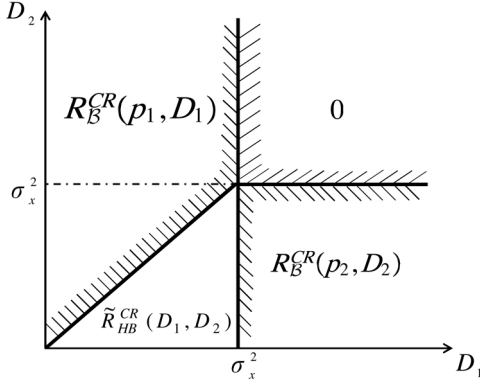


Fig. 8. Illustration of the distortion regions in the rate-distortion function (25) for a binary source with degraded erased side information and Hamming distortion.

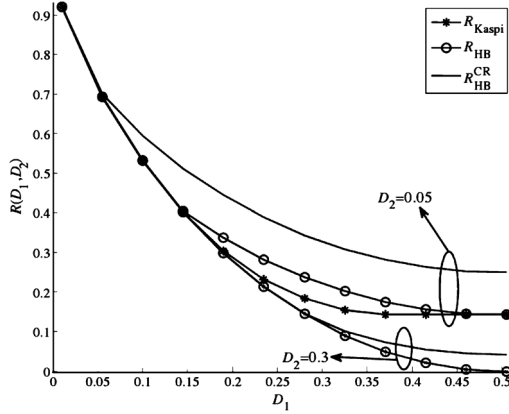


Fig. 9. Rate-distortion functions $R_{\text{Kaspi}}(D_1, D_2)$ [17], $R_{\text{HB}}(D_1, D_2)$ [12] and $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ (25) for a binary source under erased side information versus distortion D_1 ($p_1 = 1$, $p_2 = 0.35$, $D_2 = 0.05$, and $D_2 = 0.3$).

and thus the position of the erasures. For this case, the rate-distortion function $R_{\text{Kaspi}}(D_1, D_2)$ for the example at hand was calculated in [17]. Note that in the Kaspi model, the CR constraint does not affect the rate-distortion performance since the encoder has all the information available at the decoders. The second model of interest is the standard HB setting with no CR constraint, whose rate-distortion function $R_{\text{HB}}(D_1, D_2)$ for the example at hand was derived in [12]. The third model is the HB setup with CR studied here. We clearly have the inequalities

$$R_{\text{Kaspi}}(D_1, D_2) \leq R_{\text{HB}}(D_1, D_2) \leq R_{\text{HB}}^{\text{CR}}(D_1, D_2) \quad (29)$$

where the first inequality in (29) accounts for the impact of the availability of the side information at the encoder, while the second reflects the potential performance loss due to the CR constraint.

Fig. 9 shows the aforementioned rate-distortion functions with $p_1 = 1$ and $p_2 = 0.35$, which corresponds to the case where Decoder 1 has no side information, for two values of the distortion D_2 versus the distortion D_1 . For $D_2 \geq \frac{p_2}{2} = 0.175$, the given settings reduce to one in which the encoder needs to communicate information only to Decoder 1. Since Decoder 1 has no side information, the Kaspi and HB settings yield equal

performance, i.e., $R_{\text{Kaspi}}(D_1, D_2) = R_{\text{HB}}(D_1, D_2)$. Moreover, if D_1 is sufficiently smaller than D_2 , the operation of the encoder is limited by the distortion requirements of Decoder 1. In this case, Decoder 2 can in fact reconstruct as $\hat{X}_1 = \hat{X}_2$ while still satisfying its distortion constraints. Therefore, we obtain the same performance in all of the three settings, i.e., $R_{\text{Kaspi}}(D_1, D_2) = R_{\text{HB}}(D_1, D_2) = R_{\text{HB}}^{\text{CR}}(D_1, D_2)$. We also note the general performance loss due to the CR constraint, unless, as discussed above, distortion D_1 is sufficiently smaller than D_2 .

III. HB PROBLEM WITH COOPERATIVE DECODERS

The system model for the HB problem with CR and decoder cooperation is similar to the one provided in Section II-A with the following differences. Here, in addition to the encoding function given in (4) which maps the source sequence X^n into a message J_1 of nR_1 bits, there is an encoder at Decoder 1 given by

$$g_1: [1, 2^{nR_1}] \times \mathcal{Y}_1^n \rightarrow [1, 2^{nR_2}] \quad (30)$$

which maps message J_1 and the source sequence Y_1^n into a message J_2 . Moreover, instead of the decoding function given in (5), we have the decoding function for Decoder 2

$$h_2: [1, 2^{nR_1}] \times [1, 2^{nR_2}] \times \mathcal{Y}_2^n \rightarrow \hat{\mathcal{X}}_2^n \quad (31)$$

which maps the messages J_1 and J_2 and the side information Y_2^n into the estimated sequence \hat{X}_2^n .

A. Rate-Distortion Region for $X - Y_1 - Y_2$

In this section, a single-letter characterization of the rate-distortion region is derived under the assumption that the joint pmf $p(x, y_1, y_2)$ is such that the Markov chain condition $X - Y_1 - Y_2$ holds.³

Proposition 4: The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the HB source coding problem with CR and cooperative decoders under the assumption $X - Y_1 - Y_2$ is given by the union of all rate pairs (R_1, R_2) that satisfy the conditions

$$R_1 \geq I(X; \hat{X}_1 \hat{X}_2 | Y_1) \quad (32a)$$

$$\text{and } R_1 + R_2 \geq I(X; \hat{X}_2 | Y_2) + I(X; \hat{X}_1 | Y_1, \hat{X}_2) \quad (32b)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, \hat{x}_1, \hat{x}_2) = p(x, y_1)p(y_2 | y_1)p(\hat{x}_1, \hat{x}_2 | x) \quad (33)$$

for some pmf $p(\hat{x}_1, \hat{x}_2 | x)$ such that the constraints (13) are satisfied.

The proof of the converse can be easily established following cut-set arguments for bound (32a), while the bound (32b) on the sum-rate $R_1 + R_2$ can be proved following the same step as in Appendix A and substituting J with (J_1, J_2) . As for the achiev-

³Note that, unlike the conventional HB problem studied in Section II, the rate-distortion region with cooperative decoders depends on the joint distribution of the variables (Y_1, Y_2) , and thus, stochastic and physical degradedness of the side information sequences lead to different results.

ability, it follows as a straightforward extension of [8, Sec. III] to the setup at hand where Decoder 2 has side information as well. It is worth emphasizing that the reconstruction \hat{X}_2 for the Decoder 2, which has degraded side information, is conveyed by using both the direct link from the Encoder, of rate R_1 , and the path Encoder-Decoder 1-Decoder 2. The latter path leverages the better side information at Decoder 1 and the cooperative link of rate R_2 .

Remark 7: If we remove the CR constraint, the problem of determining the rate-distortion region for the setting of Fig. 3 under the Markov assumption $X - Y_1 - Y_2$ is still open. In [8], inner and outer bounds are obtained to the rate-distortion region, for the case in which the side information Y_2 is absent. The bounds were shown to coincide for the case in which Decoder 1 wishes to recover X losslessly (i.e., $D_1 = 0$) and also for certain distortion regimes in the quadratic Gaussian case. Moreover, the rate-distortion tradeoff is completely characterized in [8] for the case in which the encoder also has access to the side information. We note that, as per the discussion in Section II-D, these latter results immediately carry over to the case with the CR constraint since the encoder is informed about the side information.

Remark 8: To understand why imposing the CR constraint simplifies the problem of obtaining a single-letter characterization of the rate-distortion function, let us consider the degrees of freedom available at Decoder 1 in Fig. 3 for the use of the link of rate R_2 . In general, Decoder 1 can follow two possible strategies: the first is forwarding, whereby Decoder 1 simply forwards some of the bits received from the encoder to Decoder 2; while the second is recompression, whereby the data received from the encoder are combined with the available side information Y_1^n , compressed to at most R_2 bits per symbol, and then sent to Decoder 2. It is the interplay and contrast between these two strategies that makes the general problem hard to solve. In particular, while the strategies of forwarding/recompression and combinations thereof appear to be natural candidates for the problem, finding a matching converse when both such degrees of freedom are permissible at the decoder is difficult (see, e.g., [20]). However, under the CR constraint, the strategy of recompression becomes irrelevant, since any information about the side information Y_1^n that is not also available at the encoder cannot be leveraged by Decoder 2 without violating the CR constraint. This restriction in the set of available strategies for Decoder 1 makes the problem easier to address under the CR constraint.

B. Rate-Distortion Region for $X - Y_2 - Y_1$

In this section, a single-letter characterization of the rate-distortion region is derived under the assumption that the joint pmf $p(x, y_1, y_2)$ is such that the Markov chain relationship $X - Y_2 - Y_1$ holds.

Proposition 5: The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the HB source coding problem with CR and cooperative decoders under the assumption of the Markov chain relationship

$X - Y_2 - Y_1$ is given by the union of all rate pairs (R_1, R_2) that satisfy the conditions

$$R_1 \geq I(X; \hat{X}_1|Y_1) + I(X; \hat{X}_2|Y_2, \hat{X}_1) \quad (34a)$$

$$\text{and } R_2 \geq 0 \quad (34b)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, \hat{x}_1, \hat{x}_2) = p(x, y_2)p(y_1|y_2)p(\hat{x}_1, \hat{x}_2|x) \quad (35)$$

for some pmf $p(\hat{x}_1, \hat{x}_2|x)$ such that the constraints (13) are satisfied.

The proof of achievability follows immediately by neglecting the link of rate R_2 and using rate R_1 as per the HB scheme of Proposition 1. The converse follows by considering an enhanced system in which Decoder 2 is provided with the side information of Decoder 1. In this system, link R_2 becomes useless since Decoder 2 possesses all the information available at Decoder 1. It follows that the system reduces to the HB problem with degraded sources studied in the previous section and the bound (34a) follows immediately from Proposition 1.

Remark 9: In the case without CR, the rate-distortion function is given similarly to (34), but with the HB rate-distortion function (15) in lieu of the rate-distortion function of the HB problem with CR in (34a).

IV. CASCADE SOURCE CODING WITH CR

In this section, we first detail the system model in Fig. 4 of cascade source coding with CR. As mentioned in Section I, the motivation for studying this class of models comes from multihop applications. Next, the characterization of the corresponding rate-distortion performance is presented under the assumption that one of the two side information sequences is a degraded version of the other. Finally, following the previous section, three specific examples are worked out, namely Gaussian sources under quadratic distortion (see Section IV-C1), and binary sources with side information subject to erasures under Hamming or erasure distortion (see Section IV-C2).

A. System Model

In this section, the system model for the cascade source coding problem with CR is detailed similar to Section II-A. The problem is defined by the pmf $p_{XY_1Y_2}(x, y_1, y_2)$ and discrete alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, \hat{\mathcal{X}}_1$, and $\hat{\mathcal{X}}_2$ as follows. The source sequence X^n and side information sequences Y_1^n and Y_2^n , with $X^n \in \mathcal{X}^n$, $Y_1^n \in \mathcal{Y}_1^n$, and $Y_2^n \in \mathcal{Y}_2^n$, are such that the tuples (X_i, Y_{1i}, Y_{2i}) for $i \in [1, n]$ are i.i.d. with joint pmf $p_{XY_1Y_2}(x, y_1, y_2)$. Node 1 measures a sequence X^n and encodes it into a message J_1 of nR_1 bits, which is delivered to Node 2. Node 2 estimates a sequence $\hat{X}_1^n \in \hat{\mathcal{X}}_1^n$ within given distortion requirements. Node 2 also encodes the message J_1 received from Node 1 and the sequence Y_1^n into a message J_2 of nR_2 bits, which is delivered to Node 3. Node 3 estimates a sequence $\hat{X}_2^n \in \hat{\mathcal{X}}_2^n$ within given distortion requirements. Distortion and CR requirements are defined as in Section II-A, leading to the following definition.

Definition 2: An $(n, R_1, R_2, D_1, D_2, \epsilon)$ code for the cascade source coding problem with CR consists an encoding function for Node 1

$$g_1: \mathcal{X}^n \rightarrow [1, 2^{nR_1}] \quad (36)$$

which maps the source sequence X^n into a message J_1 ; an encoding function for Node 2

$$g_2: [1, 2^{nR_1}] \times \mathcal{Y}_1^n \rightarrow [1, 2^{nR_2}] \quad (37)$$

which maps the source sequence Y_1^n and message J_1 into a message J_2 ; a decoding function for Node 2

$$h_1: [1, 2^{nR_1}] \times \mathcal{Y}_1^n \rightarrow \hat{\mathcal{X}}_1^n \quad (38)$$

which maps message J_1 and the side information Y_1^n into the estimated sequence \hat{X}_1^n ; a decoding function for Node 3

$$h_2: [1, 2^{nR_2}] \times \mathcal{Y}_2^n \rightarrow \hat{\mathcal{X}}_2^n \quad (39)$$

which maps message J_2 and the side information Y_2^n into the estimated sequence \hat{X}_2^n ; and two encoder reconstruction functions as in (7), which map the source sequence into estimated sequences $\psi_1(X^n)$ and $\psi_2(X^n)$ at Node 1; such that the distortion constraints (8) and (9) are satisfied.

Given a distortion pair (D_1, D_2) , a rate pair (R_1, R_2) is said to be achievable if, for any $\epsilon > 0$ and sufficiently large n , there exists an $(n, R_1, R_2, D_1 + \epsilon, D_2 + \epsilon, \epsilon)$ code. The rate-distortion region $\mathcal{R}(D_1, D_2)$ is defined as the closure of all rate pairs (R_1, R_2) that are achievable given the distortion pair (D_1, D_2) .

B. Rate-Distortion Region for $X - Y_1 - Y_2$

In this section, a single-letter characterization of the rate-distortion region is derived under the assumption that the joint pmf $p(x, y_1, y_2)$ is such that the Markov chain relationship $X - Y_1 - Y_2$ holds.⁴

Proposition 6: The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the cascade source coding problem with CR is given by the union of all rate pairs (R_1, R_2) that satisfy the conditions

$$R_1 \geq I(X; \hat{X}_1 \hat{X}_2 | Y_1) \quad (40a)$$

$$\text{and } R_2 \geq I(X; \hat{X}_2 | Y_2) \quad (40b)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, \hat{x}_1, \hat{x}_2) = p(x, y_1)p(y_2|y_1)p(\hat{x}_1, \hat{x}_2|x) \quad (41)$$

for some pmf $p(\hat{x}_1, \hat{x}_2|x)$ such that the constraints (13) are satisfied.

The proof of the converse is easily established following cut-set arguments. To prove achievability, it is sufficient to consider a scheme based on binning at Node 1 and decode and

rebin at Node 2 (see [11]). Specifically, Node 1 randomly generates a standard lossy source code \hat{X}_1^n for the source X^n with rate $I(X; \hat{X}_1)$ bits per source symbol. Random binning is used to reduce the rate to $I(X; \hat{X}_1 | Y_1)$. Node 1 then maps the source X^n into the reconstruction sequence \hat{X}_2^n using a codebook that is generated conditional on \hat{X}_1^n with rate $I(X; \hat{X}_2 | \hat{X}_1)$ bits per source symbol. Using the side information Y_1^n available at Node 2, random binning is again used to reduce the rate to $I(X; \hat{X}_2 | Y_1 \hat{X}_1)$. The codebook of \hat{X}_2^n is also randomly binned to the rate $I(X; \hat{X}_2 | Y_2)$. Node 2, having recovered \hat{X}_2^n , forwards the corresponding bin index to Node 3. The latter, by choice of the binning rate, is able to obtain \hat{X}_2^n . Note that, since the reconstruction sequences \hat{X}_1^n and \hat{X}_2^n are generated by the encoder, functions ψ_1 and ψ_2 that guarantee the CR constraints (9) exist by construction.

Remark 10: If we remove the CR constraint, the problem of determining the rate-distortion region for the setting of Fig. 4 under the Markov condition $X - Y_1 - Y_2$ is still open. In the special case in which $Y_1 = Y_2$ the problem has been solved in [10] for Gaussian sources under quadratic distortion and in [12] for binary sources with erased side information under Hamming distortion.

Remark 11: Following Remark 3, if both side information sequences are causal, it can be shown that they have no impact on the rate-distortion function (40). Therefore, the rate-distortion region follows immediately from the results in (40) by removing both of the side information terms. Note that with causal side information sequences, the rate-distortion function holds for any joint pmf $p(x, y_1, y_2)$ with no degradedness requirements. Moreover, if only the side information Y_2 is causal, while Y_1 is still observed noncausally, then the side information Y_2 can be neglected without loss of optimality, and the rate-distortion region follows from (40) by removing the conditioning on Y_2 .

C. Bounds on the Rate-Distortion Region for $X - Y_2 - Y_1$

In this section, outer and inner bounds are derived for the rate-distortion region under the assumption that the joint pmf $p(x, y_1, y_2)$ is such that the Markov chain relationship $X - Y_2 - Y_1$ holds. The bounds are then shown to coincide in Section IV-C1 for Gaussian sources and in Section IV-C2 for binary sources with erased side information.

Proposition 7 (Outer Bound): The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the cascade source coding problem with CR is contained in the region $\mathcal{R}_{\text{out}}^{\text{CR}}(D_1, D_2)$, which is given by the set of all rate pairs (R_1, R_2) that satisfy the conditions

$$R_1 \geq R_{\text{HB}}^{\text{CR}}(D_1, D_2) \quad (42a)$$

$$\text{and } R_2 \geq R_{X|Y_2}^{\text{CR}}(D_2) \quad (42b)$$

where $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ is defined in (11) and we have $R_{X|Y_2}^{\text{CR}}(D_2) = \min I(X; \hat{X}_2 | Y_2)$, where the minimization is performed with respect to the conditional pmf $p(\hat{x}_2|x)$ under the distortion constraints (13) for $j = 2$.

⁴As for the HB problem with cooperative decoders studied in Section III, the rate-distortion region of the cascade source coding problem depends on the joint distribution of the variables (Y_1, Y_2) , and thus, stochastic and physical degradedness of the side information sequences lead to different results.

Proposition 8 (Inner Bound): The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the cascade source coding problem with CR contains the region $\mathcal{R}_{\text{in}}^{\text{CR}}(D_1, D_2)$, which is given by the union of all rate pairs (R_1, R_2) that satisfy the conditions

$$R_1 \geq I(X; \hat{X}_1|Y_1) + I(X; \hat{X}_2|Y_2 \hat{X}_1) \quad (43a)$$

$$\text{and } R_2 \geq I(X; \hat{X}_1|Y_2) + I(X; \hat{X}_2|\hat{X}_1 Y_2) \quad (43b)$$

$$= I(X; \hat{X}_1 \hat{X}_2|Y_2) \quad (43c)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, \hat{x}_1, \hat{x}_2) = p(x, y_2)p(y_1|y_2)p(\hat{x}_1, \hat{x}_2|x) \quad (44)$$

for some pmf $p(\hat{x}_1, \hat{x}_2|x_1)$ such that the distortion constraints (13) are satisfied.

The outer bound in Proposition 7 follows immediately from cut-set arguments similar to those in [10] and [12]. As for the inner bound of Proposition 19, the strategy works as follows. Node 1 sends the description \hat{X}_1^n to Node 2 using binning with rate $I(X; \hat{X}_1|Y_1)$. It also maps the sequence X^n into the sequence \hat{X}_2^n using a conditional codebook with respect to \hat{X}_1^n , which is binned in order to leverage the side information Y_2^n at Node 3 with rate $I(X; \hat{X}_2|\hat{X}_1, Y_2)$. Node 2 recovers \hat{X}_1^n , whose codebook is then binned to rate $I(X; \hat{X}_1|Y_2)$. Then, it forwards the so obtained bin index for \hat{X}_1^n and the bin index for the codebook of \hat{X}_2^n produced by Node 1 to Node 3. By the choice of the rates, the latter can recover both \hat{X}_1^n and \hat{X}_2^n . Since both descriptions are produced by Node 1, the CR constraint is automatically satisfied.

The inner and outer bounds defined above do not coincide in general. However, in the next sections, we provide two examples in which they coincide and thus characterize the rate-distortion region of the corresponding settings.

Remark 12: Without the CR constraint, the problem of deriving the rate-distortion region for the setting at hand under the Markov chain condition $X - Y_2 - Y_1$ is open. The problem has been solved in [10] for Gaussian sources under quadratic distortion and in [12] for binary sources with erased side information under Hamming distortion for $Y_1 = Y_2$.

1) Gaussian Sources and Quadratic Distortion: In this section, we assume the Gaussian sources in (17) and the quadratic distortion as in Section II-C and derive the rate-distortion region for the cascade source coding problem with CR.

Proposition 9: The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the cascade source coding problem with CR for the Gaussian sources in (17) and quadratic distortion is given by (42) with $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ in (19) and $R_{X|Y_2}^{\text{CR}}(D_2) = R_G^{\text{CR}}(D_2, N_2)$ [see (18)].

The proof is given in Appendix E.

2) Binary Sources With Erased Side Information and Hamming Distortion: In this section, we assume the binary sources in Fig. 7 and Hamming distortion, as in Section II-D, and derive the rate-distortion region for the cascade source coding problem with CR.

Proposition 10: The rate-distortion region $\mathcal{R}^{\text{CR}}(D_1, D_2)$ for the cascade source coding problem with CR for the binary

sources in Fig. 7 and Hamming distortion is given by (42) with $R_{\text{HB}}^{\text{CR}}(D_1, D_2)$ in (25) and $R_{X|Y_2}^{\text{CR}}(D_2) = R_B^{\text{CR}}(D_2, p_2)$ [see (23)].

The proof is given in Appendix F.

V. HB PROBLEM WITH CONR

In this section, we revisit the HB problem and relax the CR constraint to the ConR constraint of [13]. This implies that we still adopt the code as per Definition 1, but we substitute (9) with the less stringent constraint

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} [d_{e,j}(\hat{X}_{ji}, \psi_{ji}(X^n))] \leq D_{e,j}, \quad \text{for } j = 1, 2 \quad (45)$$

where $d_{e,j}(\hat{x}_j, \hat{x}_{e,j}) : \mathcal{X}_j \times \mathcal{X}_j \rightarrow [0, D_{e,\max}]$ is a per-symbol distortion metric and we have used $\psi_{ji}(X^n)$, for $j = 1, 2$, to denote the i th letter of the vector $\psi_j(X^n) = (\psi_{j1}(X^n), \dots, \psi_{jn}(X^n))$.

Definition 3: Given a distortion tuple $(D_{e,1}, D_{e,2}, D_1, D_2)$, a rate R is said to be achievable if, for any $\epsilon > 0$ and sufficiently large n , there exists an $(n, R, D_{e,1} + \epsilon, D_{e,2} + \epsilon, D_1 + \epsilon, D_2 + \epsilon, \epsilon)$ code. The rate-distortion function $R(D_{e,1}, D_{e,2}, D_1, D_2)$ is defined as $R(D_{e,1}, D_{e,2}, D_1, D_2) = \inf\{R : \text{the tuple } (D_{e,1}, D_{e,2}, D_1, D_2) \text{ is achievable}\}$.

Note that, by setting $D_{e,j} = 0$ for $j = 1, 2$, and letting $d_{e,j}(\hat{x}_j, \hat{x}_{e,j})$ be the Hamming distortion metric (i.e., $d_{e,j}(\hat{x}_j, \hat{x}_{e,j}) = 1$ if $x \neq \hat{x}_j$ and $d_{e,j}(\hat{x}_j, \hat{x}_{e,j}) = 0$ if $x = \hat{x}_j$), we obtain a relaxed CR constraint in which the average per-symbol, rather than per-block, error probability criterion is adopted.

Remark 13: The problem at hand reduces to the one studied in [13] by setting $D_1 = D_{\max}$ and $D_{e,1} = D_{e,\max}$.

Proposition 11: If the side information Y_1 is stochastically degraded with respect to Y_2 , the rate-distortion function for the HB problem with ConR is given by

$$R_{\text{HB}}^{\text{ConR}}(D_{e,1}, D_{e,2}, D_1, D_2) = \min I(X; U_1|Y_1) + I(X; U_2|Y_2 U_1) \quad (46a)$$

$$= \min I(X; U_1 U_2|Y_2) + I(U_1; Y_2|Y_1) \quad (46b)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, u_1, u_2) = p(x, y_1, y_2)p(u_1, u_2|x) \quad (47)$$

and minimization is performed with respect to the conditional pmf $p(u_1, u_2|x)$ and the deterministic functions $\hat{x}_j(u_j, y_j) : \mathcal{U}_j \times \mathcal{Y}_j \rightarrow \mathcal{X}_j$ and $\hat{x}_{e,j}(u_j, x) : \mathcal{U}_j \times \mathcal{X} \rightarrow \mathcal{X}_{e,j}$ for $j = 1, 2$, such that the distortion constraints $\mathbb{E}[d_j(X, \hat{x}_j(U_j, Y_j))] \leq D_j$ for $j = 1, 2$, and the ConR requirements

$$\mathbb{E}[d_{e,j}(\hat{x}_j(U_j, Y_j), \hat{x}_{e,j}(U_j, X))] \leq D_{e,j}, \quad \text{for } j = 1, 2 \quad (48)$$

are satisfied. Finally, (U_1, U_2) are auxiliary random variables whose alphabet cardinalities can be constrained as $|\mathcal{U}_1| \leq |\mathcal{X}| + 4$ and $|\mathcal{U}_2| \leq (|\mathcal{X}| + 2)^2$.

The proof is given in Appendix G.

Remark 14: Proposition 11 reduces to [13, Th. 2] when setting $D_1 = D_{\max}$ and $D_{e,1} = D_{e,\max}$.

Remark 15: Similar to [13, Th. 2], it can be proved that, by setting $D_{e,1} = D_{e,2} = 0$ and letting $d_{e,j}$ be the Hamming distortion for $j = 1, 2$, the rate-distortion function (46), $R_{\text{HB}}^{\text{ConR}}(0, 0, D_1, D_2)$, reduces to the rate-distortion function with CR (11).

Remark 16: Similar to Remark 15, if $D_{e,1} = 0$ and $D_{e,2} = D_{e,\max}$, the rate-distortion function (46) is given by

$$R_{\text{HB}}^{\text{CR}}(0, D_{e,\max}, D_1, D_2) = \min I(X; \hat{X}_1 | Y_1) + I(X; U_2 | Y_2 \hat{X}_1) \quad (49)$$

where the mutual information terms are evaluated with respect to the joint pmf

$$p(x, y_1, y_2, u_2, \hat{x}_1) = p(x, y_1, y_2) p(\hat{x}_1, u_2 | x) \quad (50)$$

and minimization is performed with respect to the conditional pmf $p(\hat{x}_1, u_2 | x)$ and the deterministic functions $\hat{x}_2(u_2, y_2) : \mathcal{U}_2 \times \mathcal{Y}_2 \rightarrow \hat{\mathcal{X}}_2$ and $\hat{x}_{e,2}(u_2, x) : \mathcal{U}_2 \times \mathcal{X} \rightarrow \hat{\mathcal{X}}_{e,2}$, such that the distortion constraints $\mathbb{E}[d_1(X, \hat{X}_1)] \leq D_1$ and $\mathbb{E}[d_2(X, \hat{x}_2(U_2, Y_2))] \leq D_2$ and the ConR requirement $\mathbb{E}[d_{e,2}(\hat{x}_2(U_2, Y_2), \hat{x}_{e,2}(U_2, X))] \leq D_{e,2}$ are satisfied. It can be proved that this is also the rate-distortion function under the *partial CR* requirement that there exists a function $\psi_1(X^n)$ such that (9) holds for $j = 1$ only. Similar conclusions apply symmetrically to the case where CR and ConR requirements are imposed only on the reconstruction of Decoder 2.

Remark 17: If both side information sequences are causally available at the decoders, it can be proved that they have no impact on the rate-distortion function (46). In this case, the rate-distortion function follows immediately from the results in (46) by removing conditioning on both side information sequences. Moreover, the result can be simplified by introducing a single auxiliary random variable. Similarly, if only side information Y_1 is causal, then it can be neglected with no loss of optimality, and the results follow from (46) by removing the conditioning on Y_1 .

Remark 18: We note that the ConR formulation studied in this section is more general than the conventional formulation with distortion constraints for the decoders only. Therefore, problems that are open with the conventional formulation, such as HB with cooperative decoders (see Section III) and cascade source coding (see Section IV), are *a fortiori* also open in the ConR setup.

VI. CONCLUDING REMARKS

The CR requirement [5] and its generalization in [13] substantially modify the problem of source coding in the presence

of side information at the decoders. From a practical standpoint, in various applications, such as transmission of medical records, CR is a design constraint. In these cases, evaluation of the rate-distortion performance under CR thus reveals the cost, in terms of transmission resources, associated with this additional requirement. From a theoretical perspective, adding the CR constraint to standard source coding problems with decoder side information proves instrumental in concluding about the optimality of various known strategies in settings in which the more general problem, without the CR constraint, is open [5]. This paper has extended these considerations from a point-to-point setting to three baseline multiterminal settings, namely the HB problem, the HB problem with cooperating decoders and the cascade problems. The optimal rate-distortion tradeoff has been derived in a number of cases and explicitly evaluated in various examples.

A general subject of theoretical interest is identifying those models for which the CR requirements enable a solution of problems that have otherwise resisted solutions. Examples include the HB and cascade source coding problems with no assumptions on side information degradedness and the one-helper lossy source coding problem.

APPENDIX A

PROOF OF PROPOSITION 1

We first observe that from Definition 1, distortion and CR constraints (8) and (9) depend only on the marginal pmfs $p(x, y_1)$ and $p(x, y_2)$, and so does the rate-distortion function. Therefore, in the proof, we can assume, without loss of generality, that the joint pmf $p(x, y_1, y_2)$ satisfies the Markov chain condition $X - Y_2 - Y_1$ so that it factors as [cf., (10)]

$$p(x, y_1, y_2) = p(x, y_2) \tilde{p}(y_1 | y_2). \quad (51)$$

Consider an $(n, R, D_1 + \epsilon, D_2 + \epsilon, \epsilon)$ code, whose existence is required for achievability by Definition 1. By the CR requirements (9), we first observe that we have the Fano inequalities

$$H(\psi_j(X^n) | h_j(g(X^n), Y_j^n)) \leq n\delta(\epsilon), \quad \text{for } j = 1, 2 \quad (52)$$

for n sufficiently large, where $\delta(\epsilon) = n\epsilon \log |\mathcal{X}| + H_b(\epsilon)$. Moreover, we can write

$$nR = H(J) \geq H(J | Y_1^n) \quad (53a)$$

$$\stackrel{(a)}{=} H(J | Y_1^n Y_2^n) + I(J; Y_2^n | Y_1^n) \quad (53b)$$

where (a) follows by the definition of mutual information. From now on, to simplify notation, we do not make explicit the dependence of ψ_j , g_j , and h_j on X^n and (J, Y_j^n) , respectively. We also define ψ_{ji} as the i th symbol of the sequence ψ_j so that $\psi_j = (\psi_{j1}, \dots, \psi_{jn})$.

The first term in (53b), $H(J|Y_1^n Y_2^n)$, can be treated as in [5, Sec. V.A.], or, more simply, we can proceed as follows:

$$H(J|Y_1^n Y_2^n) \stackrel{(a)}{=} I(J; X^n | Y_1^n Y_2^n) \quad (54a)$$

$$\stackrel{(b)}{\geq} I(h_1 h_2; X^n | Y_1^n Y_2^n) \quad (54b)$$

$$= I(h_1 h_2 \psi_1 \psi_2; X^n | Y_1^n Y_2^n) - I(\psi_1 \psi_2; X^n | Y_1^n Y_2^n h_1 h_2) \quad (54c)$$

$$\stackrel{(c)}{\geq} I(\psi_1 \psi_2; X^n | Y_1^n Y_2^n) - I(\psi_1 \psi_2; X^n | Y_1^n Y_2^n h_1 h_2) \quad (54d)$$

$$\stackrel{(d)}{=} I(\psi_1 \psi_2; X^n | Y_2^n) - H(\psi_1 \psi_2 | Y_1^n Y_2^n h_1 h_2) + H(\psi_1 \psi_2 | Y_1^n Y_2^n h_1 h_2 X^n) \quad (54e)$$

$$\stackrel{(e)}{\geq} I(\psi_1 \psi_2; X^n | Y_2^n) - n\delta(\epsilon) \quad (54f)$$

$$\stackrel{(f)}{\geq} \sum_{i=1}^n I(\psi_{1i} \psi_{2i}; X_i | Y_{2i}) - n\delta(\epsilon) \quad (54g)$$

where (a) follows because J is a function of X^n ; (b) follows since h_1 and h_2 are functions of (J, Y_1^n) and (J, Y_2^n) , respectively; (c) follows by using the Markov chain relationship $(\psi_1, \psi_2, X^n) - Y_2^n - Y_1^n$; (d) follows by the chain rule of mutual information and since mutual information is nonnegative; (e) follows by (52) and since entropy is nonnegative; and (f) follows by the chain rule for entropy, since X^n and Y_2^n are i.i.d., and due to the fact that conditioning decreases entropy.

Similarly, the second term in (53b), namely, $I(J; Y_2^n | Y_1^n)$, leads to

$$I(J; Y_2^n | Y_1^n) \stackrel{(a)}{\geq} I(h_1; Y_2^n | Y_1^n) \quad (55a)$$

$$= I(h_1 \psi_1; Y_2^n | Y_1^n) - I(\psi_1; Y_2^n | Y_1^n h_1) \quad (55b)$$

$$\stackrel{(b)}{\geq} I(\psi_1; Y_2^n | Y_1^n) - H(\psi_1 | Y_1^n h_1) + H(\psi_1 | Y_1^n Y_2^n h_1) \quad (55c)$$

$$\stackrel{(c)}{\geq} I(\psi_1; Y_2^n | Y_1^n) - n\delta(\epsilon) \quad (55d)$$

$$\stackrel{(d)}{\geq} \sum_{i=1}^n I(\psi_{1i}; Y_{2i} | Y_{1i}) - n\delta(\epsilon) \quad (55e)$$

where (a) follows because h_1 is a function of J and Y_1^n ; (b) follows by the chain rule of mutual information and since mutual information is nonnegative; (c) follows by (52) and since entropy is nonnegative; and (d) follows by the chain rule for entropy, since Y_2^n and Y_1^n are i.i.d., and due to the fact that conditioning decreases entropy. From (53b), (54g), and (55e), we then have

$$nR \geq \sum_{i=1}^n I(\psi_{1i} \psi_{2i}; X_i | Y_{2i}) + I(\psi_{1i}; Y_{2i} | Y_{1i}) - n\delta(\epsilon) \quad (56a)$$

$$\stackrel{(a)}{=} \sum_{i=1}^n I(X_i; \psi_{1i} | Y_{1i}) + I(X_i; \psi_{2i} | Y_{2i} \psi_{1i}) - n\delta(\epsilon) \quad (56b)$$

where (a) follows because of the Markov chain relationship $(\psi_{1i}, \psi_{2i}) - X_i - Y_{2i} - Y_{1i}$, for $i = 1, \dots, n$. By defining $\hat{X}_{ji} = \psi_{ji}$ with $j = 1, 2$ and $i = 1, \dots, n$, the proof is concluded as in [5].

APPENDIX B

PROOF OF PROPOSITION 2

As explained in the text, we need to focus only on the case in which $D_2 \leq D_1 \leq \sigma_x^2$. As per the discussion in Appendix A, we can assume, without loss of generality, that the Markov chain relationship $X - Y_2 - Y_1$ holds, so that

$$Y_2 = X + Z_2 \quad (57a)$$

$$\text{and } Y_1 = Y_2 + \tilde{Z}_1 \quad (57b)$$

where $\tilde{Z}_1 \sim \mathcal{N}(0, N_1)$ is independent of (X, Z_2) .

We first prove a converse. Calculating the rate-distortion function in (14) requires minimization over the pmf $p(\hat{x}_1, \hat{x}_2 | x)$ under the constraint (13). A minimizing $p(\hat{x}_1, \hat{x}_2 | x)$ exists by the Weierstrass theorem due to the continuity of the mutual information and the compactness of the set of pmfs defined by the constraint (13) [21]. Fixing one such optimizing $p(\hat{x}_1, \hat{x}_2 | x)$, the rate-distortion function (14) can be written as

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) = I(X; \hat{X}_2 | Y_2) + I(\hat{X}_1; Y_2 | Y_1). \quad (58)$$

The first term in (58), i.e., $I(X; \hat{X}_2 | Y_2)$, can be easily bounded using the approach in [5, p. 5007]. Specifically, we have

$$\begin{aligned} I(X; \hat{X}_2 | Y_2) &= h(X | Y_2) - h(X | \hat{X}_2 Y_2) \\ &= h(X | X + Z_2) - h(X - \hat{X}_2 | \hat{X}_2, \hat{X}_2 + (X - \hat{X}_2) + Z_2) \\ &= h(X | X + Z_2) - h(X - \hat{X}_2 | \hat{X}_2, (X - \hat{X}_2) + Z_2) \\ &\stackrel{(a)}{\geq} h(X | X + Z_2) - h(X - \hat{X}_2 | (X - \hat{X}_2) + Z_2) \\ &\stackrel{(b)}{\geq} \frac{1}{2} \log_2 \left(2\pi e \frac{\sigma_x^2}{1 + \frac{\sigma_x^2}{N_2}} \right) - \frac{1}{2} \log_2 \left(2\pi e \frac{D_2}{1 + \frac{D_2}{N_2}} \right) \\ &= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{\sigma_x^2 + N_2} \cdot \frac{D_2 + N_2}{D_2} \right) \end{aligned} \quad (59)$$

where (a) follows because conditioning decreases entropy; and (b) follows from the maximum conditional entropy lemma [14, p. 21], which implies that $h(E | E + Z_2) \leq \frac{1}{2} \log_2 (2\pi e \sigma_{E|E+Z_2}^2)$ with $E = X - \hat{X}_2$. In fact, we have that $\sigma_{E|E+Z_2}^2 \leq \frac{D_2}{1 + \frac{D_2}{N_2}}$, since the conditional variance $\sigma_{E|E+Z_2}^2$ is upper bounded by the linear minimum mean square error of the estimate of E given $E + Z_2$. This mean square error is given by $\frac{D_2}{1 + \frac{D_2}{N_2}}$, since we have $\mathbb{E}[E^2] \leq D_2$ and since Z_2 is independent of E due to the factorization (12) and to the independence of X and Z_2 . For the second term in (58), we instead have the following:

$$\begin{aligned} I(\hat{X}_1; Y_2 | Y_1) &= h(Y_2 | Y_1) - h(Y_2 | Y_1, \hat{X}_1) \\ &= \frac{1}{2} \log_2 \left(2\pi e \frac{N_1(N_2 + \sigma_x^2)}{N_1 + N_2 + \sigma_x^2} \right) - h(Y_2 | Y_1, \hat{X}_1). \end{aligned} \quad (60)$$

Moreover, we can evaluate

$$\begin{aligned}
h(Y_2|Y_1, \hat{X}_1) &= h(Y_2, Y_1|\hat{X}_1) - h(Y_1|\hat{X}_1) \\
&= h(Y_2|\hat{X}_1) + h(Y_1|Y_2, \hat{X}_1) - h(Y_1|\hat{X}_1) \\
&= h(Y_2|\hat{X}_1) - h(Y_2 + \tilde{Z}_1|\hat{X}_1) + h(Y_2 + \tilde{Z}_1|Y_2, \hat{X}_1) \\
&\stackrel{(a)}{=} h(Y_2|\hat{X}_1) - h(Y_2 + \tilde{Z}_1|\hat{X}_1) + \frac{1}{2} \log_2(2\pi e N_1) \quad (61)
\end{aligned}$$

where (a) follows because \tilde{Z}_1 is independent of Y_2 and of \hat{X}_1 , due to the factorization (12) and due to the independence of \tilde{Z}_1 and X . Next, we obtain a lower bound on the term $h(Y_2 + \tilde{Z}_1|\hat{X}_1)$ in (61) as a function of $h(Y_2|\hat{X}_1)$ by using the EPI [14, p. 22]. Specifically, by using the conditional version of the EPI [14, p. 22], we have

$$\begin{aligned}
2^{2h(Y_2 + \tilde{Z}_1|\hat{X}_1)} &\geq 2^{2h(Y_2|\hat{X}_1)} + 2^{2h(\tilde{Z}_1|\hat{X}_1)} \\
&\stackrel{(a)}{=} 2^{2h(Y_2|\hat{X}_1)} + 2^{2h(\tilde{Z}_1)} \\
&= 2^{2h(Y_2|\hat{X}_1)} + 2\pi e N_1 \quad (62)
\end{aligned}$$

where (a) follows because \tilde{Z}_1 is independent of \hat{X}_1 as explained above. The first two terms in (61) can thus be bounded as

$$\begin{aligned}
h(Y_2|\hat{X}_1) - h(Y_2 + \tilde{Z}_1|\hat{X}_1) &\leq h(Y_2|\hat{X}_1) - \frac{1}{2} \log_2(2^{2h(Y_2|\hat{X}_1)} + 2\pi e N_1) \\
&= \frac{1}{2} \log_2 \left(\frac{2^{2h(Y_2|\hat{X}_1)}}{2^{2h(Y_2|\hat{X}_1)} + 2\pi e N_1} \right) \\
&\stackrel{(a)}{\leq} \log_2 \left(\frac{2\pi e(D_1 + N_2)}{2\pi e(D_1 + N_2) + 2\pi e N_1} \right) \quad (63)
\end{aligned}$$

where (a) follows because $\log_2 \left(\frac{2^{2h(Y_2|\hat{X}_1)}}{2^{2h(Y_2|\hat{X}_1)} + 2\pi e N_1} \right)$ is an increasing function of $h(Y_2|\hat{X}_1)$ and $h(Y_2|\hat{X}_1) \leq \frac{1}{2} \log_2(2\pi e(D_1 + N_2))$, as can be proved by using the same approach used for the bounds (a) and (b) in (59). By substituting (63) into (61), and using the result in (60), we obtain

$$\begin{aligned}
I(\hat{X}_1; Y_2|Y_1) &\geq \frac{1}{2} \log_2 \left(2\pi e \frac{N_1(N_2 + \sigma_x^2)}{N_1 + N_2 + \sigma_x^2} \right) \\
&\quad - \frac{1}{2} \log_2 \left(2\pi e \frac{N_1(D_1 + N_2)}{D_1 + N_2 + N_1} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{(N_2 + \sigma_x^2)(D_1 + N_2 + N_1)}{(N_1 + N_2 + \sigma_x^2)(D_1 + N_2)} \right). \quad (64)
\end{aligned}$$

Finally, by substituting (59) and (64) into (58), we obtain the lower bound

$$\begin{aligned}
R_{\text{HB}}^{\text{CR}}(D_1, D_2) &\geq \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{\sigma_x^2 + N_2} \cdot \frac{D_2 + N_2}{D_2} \right) \\
&\quad + \frac{1}{2} \log_2 \left(\frac{(N_2 + \sigma_x^2)(D_1 + N_2 + N_1)}{(N_1 + N_2 + \sigma_x^2)(D_1 + N_2)} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{(\sigma_x^2 + N_1 + N_2)} \cdot \frac{(D_1 + N_1 + N_2)(D_2 + N_2)}{(D_1 + N_2)D_2} \right). \quad (65)
\end{aligned}$$

For achievability, we calculate (14) with $X = \hat{X}_2 + Q_2$ and $\hat{X}_2 = \hat{X}_1 + Q_1$, where $Q_1 \sim \mathcal{N}(0, D_1 - D_2)$ and $Q_2 \sim \mathcal{N}(0, D_2)$ are independent of each other and of $(\hat{X}_1, \tilde{Z}_1, Z_2)$. This leads to the upper bound

$$\begin{aligned}
R_{\text{HB}}^{\text{CR}}(D_1, D_2) &\leq I(X; \hat{X}_1 \hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \\
&= I(X; \hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \\
&= h(X|Y_2) - h(X|Y_2, \hat{X}_2) + h(Y_2|Y_1) - h(Y_2|Y_1, \hat{X}_1) \\
&= h(X|X + Z_2) - h(\hat{X}_2 + Q_2|\hat{X}_2 + Q_2 + Z_2, \hat{X}_2) \\
&\quad + h(X + Z_2|X + Z_2 + \tilde{Z}_1) - h(\hat{X}_1 + Q_1 \\
&\quad + Q_2 + Z_2|\hat{X}_1 + Q_1 + Q_2 + Z_2 + \tilde{Z}_1, \hat{X}_1) \\
&= h(X|X + Z_2) - h(Q_2|Q_2 + Z_2) + h(X + Z_2|X + Z_2 + \tilde{Z}_1) \\
&\quad - h(Q_1 + Q_2 + Z_2|Q_1 + Q_2 + Z_2 + \tilde{Z}_1) \\
&\stackrel{(a)}{=} \frac{1}{2} \log_2 \left(2\pi e \frac{\sigma_x^2}{1 + \frac{\sigma_x^2}{N_2}} \right) - \frac{1}{2} \log_2 \left(2\pi e \frac{D_2}{1 + \frac{D_2}{N_2}} \right) \\
&\quad + \frac{1}{2} \log_2 \left(2\pi e \frac{\sigma_x^2 + N_2}{1 + \frac{\sigma_x^2 + N_2}{N_1}} \right) \\
&\quad - \frac{1}{2} \log_2 \left(2\pi e \frac{D_1 + N_2}{1 + \frac{D_1 + N_2}{N_1}} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{(\sigma_x^2 + N_1 + N_2)} \cdot \frac{(D_1 + N_1 + N_2)(D_2 + N_2)}{(D_1 + N_2)D_2} \right) \quad (66)
\end{aligned}$$

where (a) follows using $h(A|A + B) = \frac{1}{2} \log_2 \left(2\pi e \frac{S_A}{1 + \frac{S_A}{S_B}} \right)$, for A and B being independent Gaussian sources with $A \sim \mathcal{N}(0, S_A)$ and $B \sim \mathcal{N}(0, S_B)$. By comparing (80) with (66), we complete the proof.

APPENDIX C PROOF OF (23)

Here, we prove that (2) equals (23) for the given sources. For the converse, we have that

$$\begin{aligned}
I(X; \hat{X}|Y) &= H(X|Y) - H(X|\hat{X}, Y) \\
&= p - pH(X|\hat{X}, Y \neq X) - (1 - p_2)H(X|\hat{X}, Y = X) \\
&= p - pH(X|\hat{X}, Y \neq X) \\
&= p - pH(X|\hat{X}) \\
&= p - pH(X \oplus \hat{X}|\hat{X}) \\
&\stackrel{(a)}{\geq} p - pH(X \oplus \hat{X}) \\
&\geq p - pH(D) \\
&= p(1 - H(D)) \quad (67)
\end{aligned}$$

where (a) follows because conditioning decreases entropy. Achievability follows by calculating (2) with $X = \hat{X} \oplus Q$ where $Q \sim \text{Ber}(D)$.

APPENDIX D
PROOF OF PROPOSITION 3

As explained in the text, we need to focus only on the case in which $D_2 \leq D_1 \leq 1/2$. As for Appendixes A and B, we can assume, without loss of generality, that the joint pmf of (x, y_1, y_2) factorizes as (51), as shown Fig. 7. We first prove a converse. Similar to (58), we can write the rate-distortion function (14) as

$$R_{\text{HB}}^{\text{CR}}(D_1, D_2) = I(X; \hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \quad (68)$$

where the mutual information terms are calculated with a distribution $p(\hat{x}_1, \hat{x}_2|x)$ minimizing (14) under the constraint (13). The first term in (68), i.e., $I(X; \hat{X}_2|Y_2)$, can be easily bounded by following the same steps used in the derivation of (67), leading to

$$I(X; \hat{X}_2|Y_2) \geq p_2(1 - H(D_2)). \quad (69)$$

For the second term in (68), we instead have the following:

$$\begin{aligned} I(\hat{X}_1; Y_2|Y_1) &= H(Y_2|Y_1) - H(Y_2|Y_1, \hat{X}_1) \\ &= H(Y_2|Y_1) - H(Y_2, Y_1|\hat{X}_1) + H(Y_1|\hat{X}_1) \\ &= H(Y_2|Y_1) - H(Y_2|\hat{X}_1) - H(Y_1|\hat{X}_1, Y_2) + H(Y_1|\hat{X}_1) \end{aligned} \quad (70)$$

$$\stackrel{(a)}{=} H(Y_2|Y_1) - H(Y_2|\hat{X}_1) - H(Y_1|Y_2) + H(Y_1|\hat{X}_1) \quad (72)$$

where (a) follows because of the Markov chain condition $Y_1 - Y_2 - \hat{X}_1$. The second term in the right-hand side of (72) can be evaluated as

$$\begin{aligned} H(Y_2|\hat{X}_1) &= H(Y_2, X|\hat{X}_1) - H(X|Y_2, \hat{X}_1) \\ &= H(X|\hat{X}_1) + H(Y_2|X, \hat{X}_1) - H(X|Y_2, \hat{X}_1) \\ &= H(X|\hat{X}_1) + H(Y_2|X) - p_2 H(X|Y_2 \neq X, \hat{X}_1) \\ &\quad - (1 - p_2) H(X|Y_2 = X, \hat{X}_1) \\ &\stackrel{(a)}{=} H(X|\hat{X}_1) + H(p_2) - p_2 H(X|\hat{X}_1) \\ &= H(p_2) + (1 - p_2) H(X|\hat{X}_1) \end{aligned} \quad (73)$$

where (a) follows because $H(Y_2|X) = H(p_2)$. The fourth term in the right-hand side of (72) can similarly be evaluated as

$$H(Y_1|\hat{X}_1) = H(p_1) + (1 - p_1) H(X|\hat{X}_1). \quad (74)$$

Substituting (73) and (74) into (72), we obtain

$$\begin{aligned} I(\hat{X}_1; Y_2|Y_1) &= H(p_1) + (1 - p_1) H(X|\hat{X}_1) - (H(p_2) \\ &\quad + (1 - p_2) H(X|\hat{X}_1)) \\ &\quad + H(Y_2|Y_1) - H(Y_1|Y_2) \\ &= H(p_1) - H(p_2) - (p_1 - p_2) H(X|\hat{X}_1) \\ &\quad + H(Y_2) - H(Y_1) \\ &\stackrel{(a)}{\geq} (p_1 - p_2) - (p_1 - p_2) H(D_1) \end{aligned} \quad (75)$$

where (a) follows since $H(Y_2) = H(p_2) + (1 - p_2)$ and $H(Y_1) = H(p_1) + (1 - p_1)$ and due to the inequality $H(X|\hat{X}_1) \leq H(D_1)$. Substituting (75) and (69) into (68), we obtain

$$\begin{aligned} R_{\text{HB}}^{\text{CR}}(D_1, D_2) &\geq p_2(1 - H(D_2)) + (p_1 - p_2)(1 - H(D_1)) \\ &= p_1(1 - H(D_1)) + p_2(H(D_1) - H(D_2)). \end{aligned} \quad (76)$$

For achievability, we calculate (14) with $X = \hat{X}_2 \oplus Q_2$ and $\hat{X}_2 = \hat{X}_1 \oplus Q_1$, where $Q_1 \sim \text{Ber}(D_1 * D_2)$ and $Q_2 \sim \text{Ber}(D_2)$ are independent of each other and of (\hat{X}_1, E_1, E_2) where $E_j = \mathbf{1}\{Y_j = e\}$ for $j = 1, 2$. This leads to the upper bound

$$\begin{aligned} R_{\text{HB}}^{\text{CR}}(D_1, D_2) &\leq I(X; \hat{X}_1 \hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \\ &= I(X; \hat{X}_2|Y_2) + I(\hat{X}_1; Y_2|Y_1) \\ &= H(X|Y_2) - H(X|Y_2, \hat{X}_2) + H(Y_2|Y_1) - H(Y_2|Y_1, \hat{X}_1) \\ &\stackrel{(a)}{=} p_2 - p_2 H(X|\hat{X}_2, Y_2 \neq X) - (1 - p_2) H(X|\hat{X}_2, Y_2 = X) \\ &\quad + p_1 H\left(\frac{p_2}{p_1}\right) \\ &\quad + \tilde{p}_1(1 - p_2) - p_1 H(Y_2|\hat{X}_1, Y_1 = e) \\ &\quad - (1 - p_1) H(Y_2|\hat{X}_1, Y_1 = X) \\ &\stackrel{(b)}{=} p_2 - p_2 H(X|\hat{X}_2) + p_1 H\left(\frac{p_2}{p_1}\right) + \tilde{p}_1(1 - p_2) \\ &\quad - p_1 \left(H\left(\frac{p_2}{p_1}\right) + \frac{\tilde{p}_1(1 - p_2)}{p_1} H(X|\hat{X}_1) \right) \\ &\stackrel{(c)}{=} p_2 - p_2 H(\hat{X}_2 \oplus Q_2|\hat{X}_2) + \tilde{p}_1(1 - p_2) \\ &\quad - \tilde{p}_1(1 - p_2) H(\hat{X}_1 \oplus Q_1 \oplus Q_2|\hat{X}_1) \\ &\stackrel{(d)}{=} p_2 - p_2 H(D_2) + \tilde{p}_1(1 - p_2) - \tilde{p}_1(1 - p_2) H(D_1) \\ &= p_1(1 - H(D_1)) + p_2(H(D_1) - H(D_2)) \end{aligned} \quad (77)$$

where (a) follows because $H(Y_2|Y_1) = p_1 H\left(\frac{p_2}{p_1}\right) + \tilde{p}_1(1 - p_2)$; (b) follows because $H(Y_2|\hat{X}_1, Y_1 = X) = H(X|\hat{X}_1, Y_1 = X) = 0$ and $H(Y_2|\hat{X}_1, Y_1 = e) = H\left(\frac{p_2}{p_1}\right) + \frac{\tilde{p}_1(1 - p_2)}{p_1} H(X|\hat{X}_1)$; (c) follows by using the inverse test channels $X = \hat{X}_2 \oplus Q_2$ and $\hat{X}_2 = \hat{X}_1 \oplus Q_1$; and (d) follows because $Q_2 \sim \text{Ber}(D_2)$ and $Q_1 \oplus Q_2 \sim \text{Ber}(D_1)$. By comparing (76) with (77), we complete the proof.

APPENDIX E
PROOF OF PROPOSITION 9

Here, we provide the proof of Proposition 9. To this end, we prove that for any pair (D_1, D_2) , there exists a joint distribution $p(\hat{x}_1, \hat{x}_2|x)$ such that (13) is satisfied and the conditions (43a) and (43b) coincide with (42a) and (42b), respectively. This requires that the inner and outer bounds of Propositions 7 and 8 coincide.

We distinguish the four regions in the (D_1, D_2) plane depicted in Fig. 5. If $D_1 \geq \sigma_x^2$ and $D_2 \geq \sigma_x^2$, it is enough to set $\hat{X}_1 = \hat{X}_2 = 0$ in (43) to prove. For $D_1 \leq \sigma_x^2$ and $D_2 \geq$

$\min(D_1, \sigma_x^2)$, we instead set $\hat{X}_1 = \hat{X}_2$ and $X = \hat{X}_1 + Q_1$ in (43), where $Q_1 \sim \mathcal{N}(0, D_1)$ is independent of \hat{X}_1 . Following the discussion in Section II-C, it is easy to see that this choice is such that (43) coincides with (42). Next, in the subregion where $D_1 \geq \sigma_x^2$ and $D_2 \leq \sigma_x^2$, we select $\hat{X}_1 = 0$ and $X = \hat{X}_2 + Q_2$ in (43), where $Q_2 \sim \mathcal{N}(0, D_2)$ is independent of \hat{X}_2 . Finally, for the region in Fig. 5, for which $D_2 \leq D_1 \leq \sigma_x^2$, we choose $X = \hat{X}_2 + Q_2$ and $\hat{X}_2 = \hat{X}_1 + Q_1$, where $Q_1 \sim \mathcal{N}(0, D_1 - D_2)$ and $Q_2 \sim \mathcal{N}(0, D_2)$ are independent of each other and of (\hat{X}_1, E_1, E_2) . With this choice, following the derivations in Appendix B, we conclude that condition (43a) coincides with (42a). As for (43b), we proceed as follows:

$$\begin{aligned}
& I(X; \hat{X}_1|Y_2) + I(X; \hat{X}_2|\hat{X}_1 Y_2) \\
&= I(X; \hat{X}_1 \hat{X}_2|Y_2) \\
&= h(X|Y_2) - h(X|\hat{X}_1, \hat{X}_2, Y_2) \\
&= h(X|X + Z_2) \\
&\quad - h(\hat{X}_1 + Q_1 + Q_2|\hat{X}_1, \hat{X}_1 + Q_1, \hat{X}_1 + Q_1 + Q_2 + Z_2) \\
&= h(X|X + Z_2) - h(Q_1 + Q_2|Q_1, Q_1 + Q_2 + Z_2) \\
&= h(X|X + Z_2) - h(Q_2|Q_2 + Z_2) \\
&= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{1 + \frac{\sigma_x^2}{N_2}} \right) - \frac{1}{2} \log_2 \left(\frac{D_2}{1 + \frac{D_2}{N_2}} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{\sigma_x^2 + N_2} \frac{D_2 + N_2}{D_2} \right) \\
&= R_{\mathcal{G}}^{CR}(D_2, N_2)
\end{aligned} \tag{78}$$

which concludes the proof.

APPENDIX F

PROOF OF PROPOSITION 10

Here, we provide the proof of Proposition 10. Following similar steps as in Appendix E, we prove that for any pair (D_1, D_2) , there exists a joint distribution $p(\hat{x}_1, \hat{x}_2|x)$ such that (13) is satisfied and the conditions (43a) and (43b) coincide with (42a) and (42b), respectively. This requires that the inner and outer bounds of Propositions 7 and 8 coincide.

We distinguish the four region in the (D_1, D_2) plane depicted in Fig. 8. If $D_1 \geq 1/2$ and $D_2 \geq 1/2$, it is enough to set $\hat{X}_1 = \hat{X}_2 = 0$ in (43) to prove the desired result. For $D_1 \leq 1/2$ and $D_2 \geq \min(D_1, 1/2)$, we instead set $\hat{X}_1 = \hat{X}_2$ and $X = \hat{X}_1 \oplus Q_1$ in (43), where $Q_1 \sim \text{Ber}(D_1)$ is independent of \hat{X}_1 . Following the discussion in Section II-D, it is easy to see that this choice is such that (43) coincides with (42). Next, in the subregion where $D_1 \geq 1/2$ and $D_2 \leq 1/2$, we select $\hat{X}_1 = 0$ and $X = \hat{X}_2 \oplus Q_2$ in (43), where $Q_2 \sim \text{Ber}(D_2)$ is independent of \hat{X}_2 . Finally, for the region in Fig. 8, for which $D_2 \leq D_1 \leq 1/2$, we choose $X = \hat{X}_2 \oplus Q_2$ and $\hat{X}_2 = \hat{X}_1 \oplus Q_1$, where $Q_1 \sim \text{Ber}(D_1 - D_2)$ and $Q_2 \sim \text{Ber}(D_2)$ are independent of each other and of (\hat{X}_1, E_1, E_2) . With this choice, following the derivations in Appendix D, we conclude that condition (43a) coincides with (42a). As for (43b), we proceed as follows:

$$\begin{aligned}
I(X; \hat{X}_1 \hat{X}_2|Y_2) &= H(X|Y_2) - H(X|\hat{X}_1, \hat{X}_2, Y_2) \\
&= p_2 - p_2 H(X|\hat{X}_1, \hat{X}_2, Y_2 \neq X) \\
&\quad - (1 - p_2) H(X|\hat{X}_1, \hat{X}_2, Y_2 = X) \\
&= p_2 - p_2 H(X|\hat{X}_1, \hat{X}_2) \\
&\stackrel{(a)}{=} p_2 - p_2 H(X|\hat{X}_2) \\
&= p_2 - p_2 H(\hat{X}_2 \oplus Q_2|\hat{X}_2) \\
&= p_2 - p_2 H(D_2) \\
&= R_B^{CR}(D_2, p_2)
\end{aligned} \tag{79}$$

where (a) follows by the Markov chain relationship $X - \hat{X}_2 - \hat{X}_1$. This completes the proof.

APPENDIX G

PROOF OF PROPOSITION 11

The proof of the achievability follows from standard arguments, similar to [6]. For the converse, following the proof of [6, Th. 3] we have that for any $(R, D_{e,1} + \epsilon, D_{e,2} + \epsilon, D_1 + \epsilon, D_2 + \epsilon)$ code, the following inequality holds:

$$nR \geq \sum_{i=1}^n I(X_i; U_{1i}|Y_{1i}) + I(X_i; U_{2i}|Y_{2i}) \tag{80}$$

$$D_{e,ji} = \mathbb{E}_{X^n Y_j^n} [d_{e,j}(h_{ji}^n(J, Y_j^n), \psi_{ji}(X^n))] \tag{81a}$$

$$\stackrel{(a)}{=} \mathbb{E}_{X^n U_{ji} Y_{ji}} [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \psi_{ji}(X^n))] \tag{81b}$$

$$= \mathbb{E}_{X^n U_{ji}} \mathbb{E}_{Y_{ji}} [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \psi_{ji}(X_i, X^{n \setminus i})) | X^n U_{ji}] \tag{81c}$$

$$\begin{aligned}
&= \sum_{x^n \in \mathcal{X}^n, u_{ji} \in \mathcal{U}} p(x^n, u_{ji}) \mathbb{E}_{Y_{ji}} \\
&\quad [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \psi_{ji}(X_i, X^{n \setminus i})) | X_i = x_i, X^{n \setminus i} = x^{n \setminus i}, U_{ji} = u_{ji}]
\end{aligned} \tag{81d}$$

$$\stackrel{(b)}{\geq} \sum_{x^n \in \mathcal{X}^n, u_{ji} \in \mathcal{U}} p(x^n, u_{ji}) \mathbb{E}_{Y_{ji}} [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \psi_{ji}(X_i, X^{n \setminus i})) | X_i = x_i, X^{n \setminus i} = x^{n \setminus i}(x_i, u_{ji}), U_{ji} = u_{ji}] \tag{81e}$$

$$\stackrel{(c)}{=} \sum_{x^n \in \mathcal{X}^n, u_{ji} \in \mathcal{U}} p(x^n, u_{ji}) \mathbb{E}_{Y_{ji}} [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \hat{x}_{e,ji}(U_{ji}, X_i)) | X_i = x_i, U_{ji} = u_{ji}] \tag{81f}$$

$$= \mathbb{E}_{X_i U_{ji} Y_{ji}} [d_{e,j}(\hat{x}_{ji}(U_{ji}, Y_{ji}), \hat{x}_{e,ji}(U_{ji}, X_i))] \tag{81g}$$

$$\mathbf{x}^{*n \setminus i}(x_i, u_{ji}) \in \arg\min_{\mathbf{x}^{n \setminus i} \in \mathcal{X}^{n \setminus i}} \mathbb{E}_{Y_{ji}}[d_{e,j}(\hat{\mathbf{x}}_{ji}(U_{ji}, Y_{ji}), \psi_{ji}(X_i, X_i^{n \setminus i})) | X_i = x_i, X^{n \setminus i} = \mathbf{x}^{n \setminus i}, U_{ji} = u_{ji}]$$

with the definitions $U_{ji} \triangleq (J, Y_j^{n \setminus i})$, for $j = 1, 2$, with $Y_j^{n \setminus i} = [Y_{j1}^{i-1}, Y_{j(i+1)}^n]$. Note that with the given definition of U_{ji} , we have that the i th element of the decoding functions (5)–(6) can be written as $h_{ji}(J, Y_j^n) = \hat{\mathbf{x}}_{ji}(U_{ji}, Y_{ji})$ for all $i = 1, \dots, n$ and $j = 1, 2$. Now, defining $D_{e,ji} \triangleq \mathbb{E}[d_{e,j}(h_{ji}(M, Y_j^n), \psi_{ji}(X^n))]$, we have the chain of inequalities [see (81a)–(81g) at the bottom of the previous page] for the code at hand and $j = 1, 2$: where (a) follows by using the definition of random variables $U_j = (J, Y_j^{n \setminus i})$; (b) follows by selecting $\mathbf{x}^{*n \setminus i}(x_i, u_{ji})$ as shown by the equation at the top of the page, and (c) follows from the Markov chain relationship $Y_{ji} - (X_i, U_{ji}) - X^{n \setminus i}$ and from the definition $\hat{\mathbf{x}}_{e,ji}(U_{ji}, X_i) = \psi_{ji}(X_i, \mathbf{x}^{*n \setminus i}(X_i, U_{ji}))$. Let Q be a uniform random variable over the interval $[1, n]$ and independent of the variables $(X^n, Y_1^n, Y_2^n, U_1^n, U_2^n, \hat{X}_1^n, \hat{X}_2^n, \hat{X}_{e,1}^n, \hat{X}_{e,2}^n)$ and define the random variables $U_j \triangleq (Q, U_{jQ})$, $X \triangleq X_Q$, $Y_j \triangleq Y_{jQ}$, $\hat{X}_j \triangleq \hat{X}_{jQ}$, and $\hat{X}_{e,j} \triangleq \hat{X}_{e,jQ}$ for $j = 1, 2$. Moreover, note that \hat{X}_j is a deterministic function of U_{ji} and Y_{ji} , and $\hat{X}_{e,j}$ is a deterministic function of U_{ji} and X_i for $j = 1, 2$. The proof is completed by using (45) and the fact that the term $I(X_i; U_{1i}|Y_{1i}) + I(X_i; U_{2i}|Y_{2i})$ in (80) is convex with respect to the pmf $p(u_{1i}, u_{2i}|x_i)$, using standard steps (see, e.g., [11]).

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Behzad Ahmadi (S'10) received the B.Sc. and M.Sc. degrees in electrical and computer engineering from Isfahan University of Technology, Isfahan, Iran in August 2005 and February 2008, respectively. He is currently a Ph.D. candidate in electrical and computer engineering at the Center for Wireless Communications and Signal Processing Research (CWCSRP), New Jersey Institute of Technology (NJIT), Newark, NJ. His research interests include network information theory and bio-medical signal processing. Behzad Ahmadi was a recipient of the Hashimoto Fellowship Scholarship 2011–12.

Ravi Tandon (S'03–M'09) received the B.Tech degree in electrical engineering from the Indian Institute of Technology (IIT), Kanpur in 2004 and the Ph.D. degree in electrical and computer engineering from the University of Maryland, College Park in 2010. From 2010 until 2012, he was a post-doctoral research associate with Princeton University. In 2012, he joined Virginia Polytechnic Institute and State University (Virginia Tech) at Blacksburg, where he is currently a Research Assistant Professor in the Department of Electrical and Computer Engineering. His research interests are in network information theory, communication theory for wireless networks and information theoretic security.

Dr. Tandon is a recipient of the Best Paper Award at the Communication Theory symposium at the 2011 IEEE Global Telecommunications Conference.

Osvaldo Simeone (M'02) received the M.Sc. degree (with honors) and the Ph.D. degree in information engineering from Politecnico di Milano, Milan, Italy, in 2001 and 2005, respectively.

He is currently with the Center for Wireless Communications and Signal Processing Research (CWCSRP), New Jersey Institute of Technology (NJIT), Newark, where he is an Assistant Professor. His current research interests concern the cross-layer analysis and design of wireless networks with emphasis on information-theoretic, signal processing, and queuing aspects. Specific topics of interest are: cognitive radio, cooperative communications, ad hoc, sensor, mesh and hybrid networks, distributed estimation, and synchronization.

Dr. Simeone is a corecipient of Best Paper Awards of the IEEE SPAWC 2007 and IEEE WRECOM 2007. He currently serves as an Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS.

H. Vincent Poor (S'72–M'77–SM'82–F'87) received the Ph.D. degree in electrical engineering and computer science from Princeton University in 1977. From 1977 until 1990, he was on the faculty of the University of Illinois at Urbana-Champaign. Since 1990 he has been on the faculty at Princeton, where he is the Dean of Engineering and Applied Science, and the Michael Henry Strater University Professor of Electrical Engineering. Dr. Poor's research interests are in the areas of stochastic analysis, statistical signal processing and information theory, and their applications in wireless networks and related fields including social networks and smart grid. Among his publications in these areas are *Classical, Semi-classical and Quantum Noise* (Springer, 2012) and *Smart Grid Communications and Networking* (Cambridge University Press, 2012).

Dr. Poor is a member of the National Academy of Engineering and the National Academy of Sciences, a Fellow of the American Academy of Arts and Sciences, and an International Fellow of the Royal Academy of Engineering (U. K.). He is also a Fellow of the Institute of Mathematical Statistics, the Optical Society of America, and other organizations. In 1990, he served as President of the IEEE Information Theory Society, in 2004–07 as the Editor-in-Chief of these TRANSACTIONS, and in 2009 as General Co-chair of the IEEE International Symposium on Information Theory, held in Seoul, South Korea. He received a Guggenheim Fellowship in 2002 and the IEEE Education Medal in 2005. Recent recognition of his work includes the 2010 IET Ambrose Fleming Medal for Achievement in Communications, the 2011 IEEE Eric E. Sumner Award, the 2011 IEEE Information Theory Paper Award, and honorary doctorates from Aalborg University, the Hong Kong University of Science and Technology, and the University of Edinburgh.