1. a. $K^* = \{ z \in \mathbb{R}^n \mid z^T x \geq 0 \text{ for all } x \in K \}$

$$= \{ z \in \mathbb{R}^n \mid z^T A y \geq 0 \text{ for all } y \succeq 0 \}$$

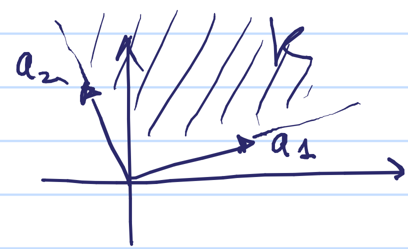
But we know that $(\mathbb{R}^k_+)^* = (\mathbb{R}^k_+)$, or equivalently

$$\{ x \in \mathbb{R}^k \mid x^T y \geq 0 \text{ for all } y \succeq 0 \} = \mathbb{R}^k_+$$

so:

$$K^* = \{ z \in \mathbb{R}^n \mid A^T z \succeq 0 \}$$

b. $n = 2$, $k = 2$



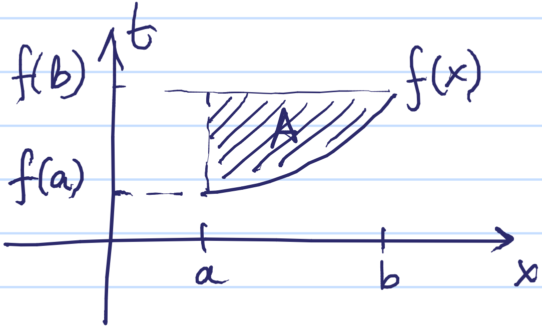
$A = [\, a_1 \mid a_2 \,]$, a_1 and a_2 columns

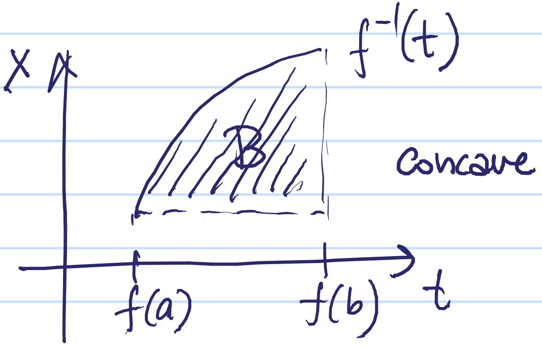
K is proper if

- closed, which is true for any A
- solid (non-empty interior)
- pointed

} true if a_1, a_2 linearly independent

2.





To see that f^{-1} is concave, note that the two shaded areas A and B are related by rotation (which preserve convexity) and that A is a convex set since it can be obtained by intersecting epi f with a halfspace. Therefore, B is a convex set. The epigraph of $-f^{-1}$, or the hypograph of f^{-1}, is hence also a convex set.

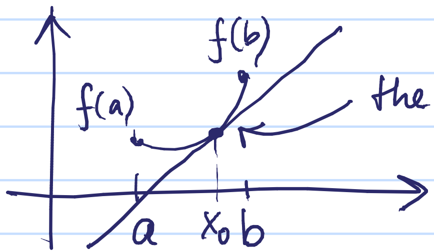
Since f^{-1} is monotonically increasing, it is both quasi-convex and quasi-concave, i.e., quasi-linear.

3. If f convex: bounded $\Rightarrow$ constant

Proof by contradiction – we prove that

If f convex: not constant $\Rightarrow$ not bounded

To this end, note that if f is not constant, we must have

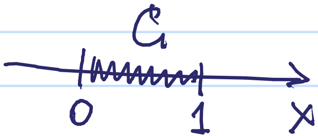


the subgradient $g \neq 0$

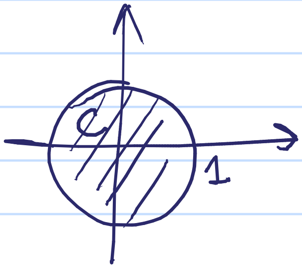
$$\Rightarrow \quad f(x) \geq g^T(x - x_0) + f(x_0)$$

$$\longrightarrow \infty$$ for a sequence x_k such that $g^T x_k \longrightarrow \infty$

4.



$$S_C(x) = \sup_{y \in C} yx = \begin{cases} x & \text{if } x \geq 0 \quad (\text{since } y^* = 1) \\ 0 & \text{if } x < 0 \quad (\text{since } y^* = 0) \end{cases}$$



$$S_C(x) = \sup_{y \in C} y^T x = \|x\|_2 \quad \left(\text{since } y^* = \frac{x}{\|x\|_2}\right)$$

5. $f(x,y) = -\log(x^2 - \|y\|_2^2)$

$$= \underbrace{-\log x}_{\text{convex}} \underbrace{- \log\Big(\underbrace{x - \frac{\|y\|_2^2}{x}}_{\text{concave (quadratic over linear)}}\Big)}_{}$$

$-\log$ is convex and $-\widetilde{\log}$ non-increasing

$\Rightarrow$ convex

6. Since $f(x)$ is quasi-linear

$C_\alpha = \{x \mid f(x) \le \alpha\}$ and $S_\alpha = \{x \mid f(x) \ge \alpha\}$

are convex. However, it is also evident that C_α and S_α partition the space $\mathbb{R}^n \Rightarrow C_\alpha$ and S_α must be half-spaces

7. $$f^*(y) = \sup_{x \geq 0} \left(yx - x^p \right)$$

- If $y \leq 0$, we clearly have $f^*(y) = 0$ since $yx - x^p \leq 0$ for all $x \geq 0$.

- If $y > 0$, then we calculate

$$\frac{d}{dx}(yx - x^p) = y - px^{p-1} = 0 \implies x = \left(\frac{y}{p}\right)^{1/p-1}$$

and hence

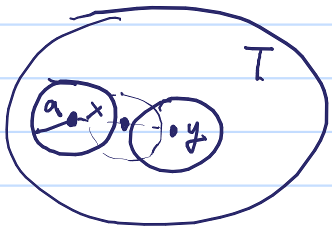
$$f^*(y) = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{p/p-1}$$

8. a. For any two points $x, y \in S$, we need to show that $\alpha x + (1-\alpha) y \in S$, that is, $\text{dist}(\alpha x + (1-\alpha) y, C) \le a$

Recall that $\text{dist}(x, S)$ is a convex function if G is a convex set and hence

$$\text{dist}(\alpha x + (1-\alpha) y, C) \le \alpha\, \text{dist}(x, C) + (1-\alpha)\, \text{dist}(y, C)$$
$$\le a$$

b. For any two points $x, y \in T$, we need to show that $\alpha x + (1-\alpha) y \in T$, that is,



$$B(\alpha x + (1-\alpha) y, a) \subseteq G$$

$$\Leftrightarrow \quad \alpha x + (1-\alpha) y + u \in G \text{ for all } \|u\| \le a$$

$$\Leftrightarrow \quad \alpha x + (1-\alpha) y + \alpha u + (1-\alpha) u \in G$$

$$\Leftrightarrow \quad \alpha \underbrace{(x+u)}_{\in G} + (1-\alpha) \underbrace{(y+u)}_{\in G \text{ since } x, y \in T} \in G$$

which is true by convexity of G.